Symmetries in Subatomic Systems [Guidance for Theory and Experiment]

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and
University of Strasbourg, F-67037 Strasbourg, FRANCE

September 27, 2010

Motto: Schools are for students while experts prefer conferences

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• All teachers will be willing to discuss physics (almost) any time!!

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- These subjects will be presented in an elementary way so that they can be self-understandable, after the download - as an introduction or a reminder

Underlying Common Themes of the EJC 2010

... listed in an arbitrary order ...

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Symmetries and Symmetry Violation in Relation to:

1. Nucleon structure; quark degrees of freedom; C, CP, T-violation

... listed in an arbitrary order ...

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Symmetries and Symmetry Violation in Relation to:

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6. Reactions, dynamics, associated symmetry issues

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In this presentation:

Lesson 1. From Molecular to Subatomic Symmetries
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Lesson 6. From Nuclear to Sub-Nuclear Particles,
Their Degrees of Freedom and Symmetries

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Lesson 7. The Heaviest Quarks

Lesson 8. Quantum Relativistic Wave Equation, Related Symmetries

Lesson 9. A Short Descriptive Introduction to QCD

Ecole Joliot–Curie 2010

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Nucleons

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Nuclei

Symmetries in Subatomic Systems

Nucleons | **Nuclei** | **Atoms**

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Consequences for Experiment: Very diverse methods

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Consequences for Theory:

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Consequences for Theory: Diverse Concepts and Methods

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Consequences for the teacher?

Symmetries in Subatomic Systems

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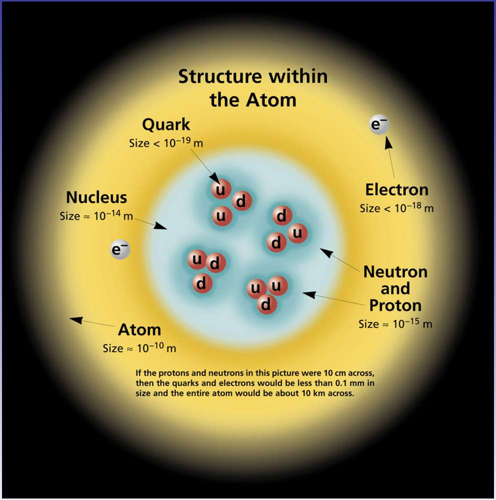
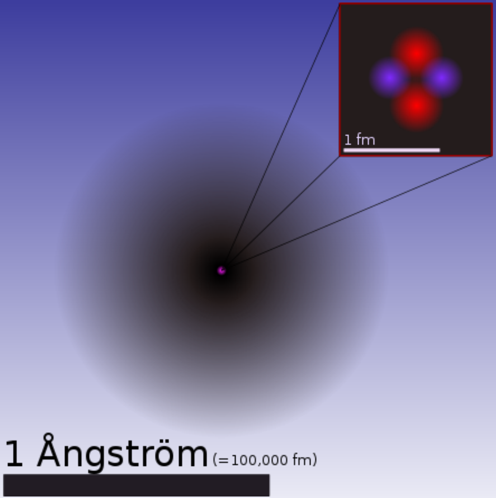
For that we will need a separate overlay!!!

Let Us Be More Precise about these Sizes...



Let Us Be More Precise: This Time He-Atom



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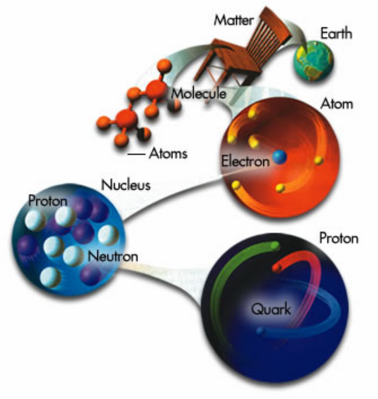

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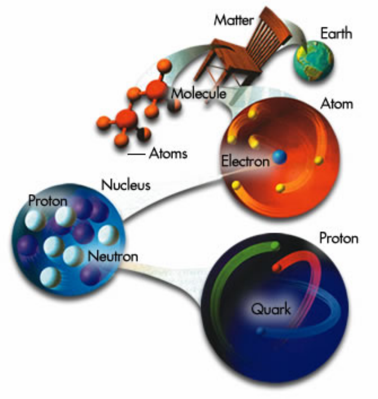

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In other words: Those who wish to learn - please download the lectures, re-read and discuss with the colleagues and your teachers!

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At the beginning of our *International School* I would like to recall a few important pages from the history of the *French research* related in particular to the Curie 'dynasty' (the capital, Alsace and Poland)

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> *The Nobel Prize in Physics 1903 was divided, one half awarded to Antoine Henri Becquerel, the other half jointly to Pierre Curie and Marie Curie, née Skłodowska.*

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Radium Institute, Paris
(Inaugurated in 1919)

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Radioactivity: From Hangar of Marie-Pierre Curie...

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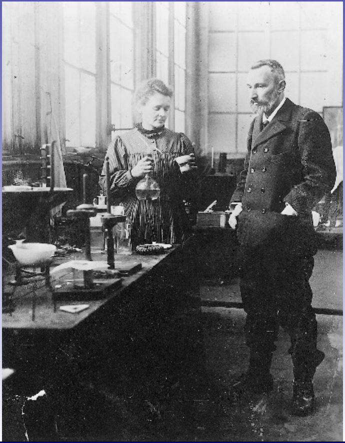

Ecole de physique et chimie:

Hangar where Maria and Pierre worked on their discoveries

Left: Interior of their laboratory

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A Few Words About History: The School

- Between 1982 and 2008 the *Ecoles Joliot-Curie* are in French ...
- ... whereas the present School is its *Second International Edition*

THE SUPPORTING ORGANISATIONS:

- Institut National de Physique Nucléaire et de Physique des Particules
- Direction des Sciences de la Matiere du CEA
- ISOLDE - (CERN, Genève)
- Service de Physique Nucléaire du CEA/DAM

SYMMETRIES in SUBATOMIC SYSTEMS

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Symmetry

SYMMETRIES in SUBATOMIC SYSTEMS

[Ecole Joliot-Curie 2010]

Symmetry - As a Matter of Fact:

SYMMETRIES in SUBATOMIC SYSTEMS

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Symmetry - As a Matter of Fact: What Is It?

The Word: *Symmetry*, Its Origin and Meaning

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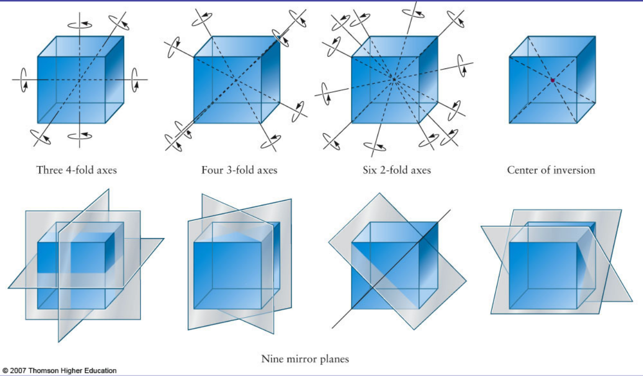
- Observe that today's meaning is less general as compared to the original sense! Today's meaning is merely a particular case!

Groups as a Language - Posing the Problem

- Do the symmetry-operations illustrated form the full set of symmetries?



- Observe a fundamental property: Combination of a symmetry operation with another symmetry operation is yet another symmetry operation !!!

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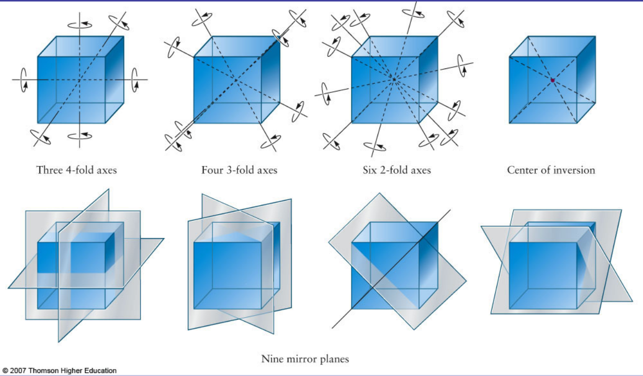

- Denoting symmetry operations S_1, S_2, $\ldots$ $S_f \equiv \{S\}$ we must have

$$\boxed{\forall\ i,j\ \ \exists\ k\ \text{ such that }\ S_i \circ S_j = S_k \in \{S\}}$$

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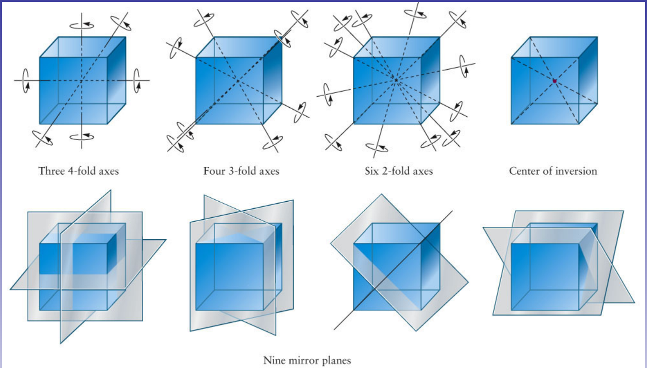

- Moreover - transforming nothing is also a symmetry operation $E \in \{S\}$

$$\boxed{\forall\ i \quad S_i \circ E = S_i \in \{S\}}$$

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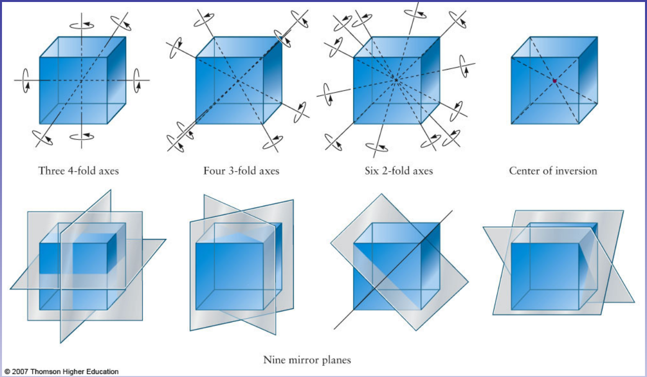
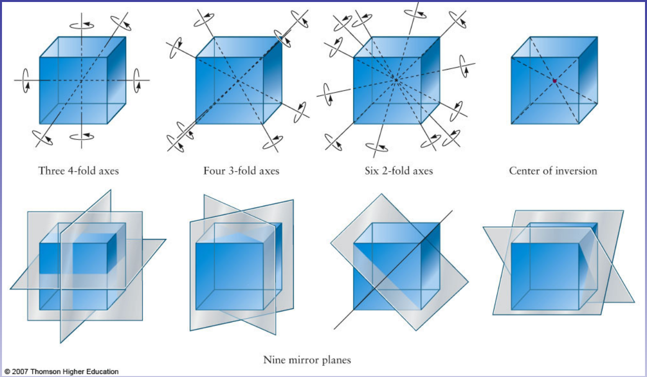


- If we can rotate (reflect) from left to right - why not from right to left?

$$\boxed{\forall\; S_i \;\; \exists \;\; S_i^{-1} \;\; \text{such that} \;\; S_i \circ S_i^{-1} = E \in \{S\}}$$

Groups as a Language - Posing the Problem

- Do the symmetry-operations illustrated form the full set of symmetries?



- Motivated by mathematical pedantry we can also verify easily that:

$$\boxed{\forall\ i,j,k\ \ (S_i \circ S_j) \circ S_k = S_i \circ (S_j \circ S_k)}$$

The Formal Group Definition

- Any set of symmetry transformations must satisfy the properties

$$\boxed{\forall\ \hat{\mathbf{g}}_1 \text{ and } \hat{\mathbf{g}}_2 \in \{\hat{\mathbf{g}}\}: \quad \hat{\mathbf{g}}_1 \circ \hat{\mathbf{g}}_2 = \hat{\mathbf{g}} \in \{\hat{\mathbf{g}}\}} \tag{1}$$

$$\boxed{\forall\ \hat{\mathbf{g}} \in \{\hat{\mathbf{g}}\}: \quad \hat{\mathbf{g}} \circ \hat{\mathbf{e}} = \hat{\mathbf{g}} \quad \text{and} \quad \hat{\mathbf{e}} \circ \hat{\mathbf{g}} = \hat{\mathbf{g}}} \tag{2}$$

$$\boxed{\forall\ \hat{\mathbf{g}} \in \{\hat{\mathbf{g}}\}:\ \exists\ \hat{\mathbf{g}}' \in \{\hat{\mathbf{g}}\} \quad \to \quad \hat{\mathbf{g}}' \circ \hat{\mathbf{g}} = \hat{\mathbf{e}}} \tag{3}$$

$$\boxed{\forall\ \hat{\mathbf{g}} \in \{\hat{\mathbf{g}}\}: \hat{\mathbf{g}}_1 \circ [\hat{\mathbf{g}}_2 \circ \hat{\mathbf{g}}_3] = [\hat{\mathbf{g}}_1 \circ \hat{\mathbf{g}}_2] \circ \hat{\mathbf{g}}_3} \tag{4}$$

- In mathematics any set with the above properties is called a group

A Lesson that Some Others Learned [?]

B. Adams et al. in *"Lie Algebraic Methods and Their Applications"* observe, not without a certain dose of satisfaction$^{*)}$:

> *"We wish to recall that the group theoretical methods, although greatly developed by pioneers of quantum mechanics [Wigner 1931, Weyl 1931], were subsequently in abeyance for a long time. In fact a certain number of physicists expressed a certain proudness when claiming that they can well* get along without it *[Condon and Shortley, 1935; pp.10-11], and the approach was even referred to as* group pest *by some."*

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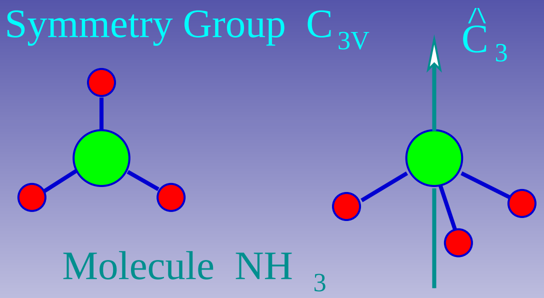
*)In *"Dynamical Groups and Spectrum Generating Algebras"*,
Eds. Arno Böhm, Yuval Ne'eman, Asim Orhan Barut

Part I

From Molecular to Subatomic Symmetries [Illustrations of Mathematics & Physics]

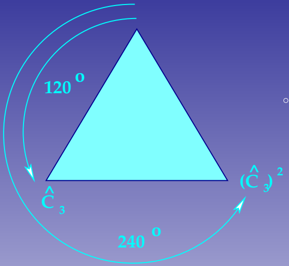
Symmetry Groups - An Elementary Example

- The NH_3 (ammonia) molecule has the symmetry group 'of the triangle'



Ammonia molecule, NH_3, is an example of the $\mathcal{C}_{3v}$ symmetry in molecular quantum chemistry. The nitrogen atom represented by a big circle lies off the plane defined by the three hydrogen atoms.

Symmetry Groups - An Elementary Example



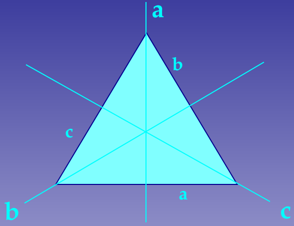


Illustration of the symmetry rotations for the $\mathcal{C}_{3v}$ group, left-hand side and the plane-reflections, right-hand side. Lines passing through the centre of the triangle represent reflections $\hat{\sigma}_{v_a}$, $\hat{\sigma}_{v_b}$ and $\hat{\sigma}_{v_c}$.

Symmetry Groups - How To Remember Them

- Observe that combinations of reflections give rotations [low-right]

$\mathcal{C}_{3v}$	e	$\hat{\mathcal{C}}_3$	$\hat{\mathcal{C}}_3^2$	$\hat{\sigma}_{v_a}$	$\hat{\sigma}_{v_b}$	$\hat{\sigma}_{v_c}$
e	e	$\hat{\mathcal{C}}_3$	$\hat{\mathcal{C}}_3^2$	$\hat{\sigma}_{v_a}$	$\hat{\sigma}_{v_b}$	$\hat{\sigma}_{v_c}$
$\hat{\mathcal{C}}_3$	$\hat{\mathcal{C}}_3$	$\hat{\mathcal{C}}_3^2$	e	$\hat{\sigma}_{v_c}$	$\hat{\sigma}_{v_a}$	$\hat{\sigma}_{v_b}$
$\hat{\mathcal{C}}_3^2$	$\hat{\mathcal{C}}_3^2$	e	$\hat{\mathcal{C}}_3$	$\hat{\sigma}_{v_b}$	$\hat{\sigma}_{v_c}$	$\hat{\sigma}_{v_a}$
$\hat{\sigma}_{v_a}$	$\hat{\sigma}_{v_a}$	$\hat{\sigma}_{v_b}$	$\hat{\sigma}_{v_c}$	e	$\hat{\mathcal{C}}_3$	$\hat{\mathcal{C}}_3^2$
$\hat{\sigma}_{v_b}$	$\hat{\sigma}_{v_b}$	$\hat{\sigma}_{v_c}$	$\hat{\sigma}_{v_a}$	$\hat{\mathcal{C}}_3^2$	e	$\hat{\mathcal{C}}_3$
$\hat{\sigma}_{v_c}$	$\hat{\sigma}_{v_c}$	$\hat{\sigma}_{v_a}$	$\hat{\sigma}_{v_b}$	$\hat{\mathcal{C}}_3$	$\hat{\mathcal{C}}_3^2$	e

Example of a Group Multiplication Table: Here the $\mathcal{C}_{3v}$-Group

Another Quick Morning Quiz

- **What is the molecular interpretation of the symmetry groups:**

$$\begin{aligned}\mathcal{G}_1 &= \{e, \hat{C}_2^z, \hat{C}_2^y, \hat{C}_2^x\} = ?\\ \mathcal{G}_2 &= \{e, \hat{C}_2^z, \hat{\sigma}_v, \hat{\sigma}_v'\} = ?\\ \mathcal{G}_3 &= \{e, \hat{C}_2^z, \hat{\mathcal{I}}, \hat{\sigma}_h\} = ?\end{aligned}$$

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- Consider a four-element group with the generic structure

$$\mathbf{G = \{e, g_1, g_2, g_3\}} \quad \text{with} \quad \mathbf{g_1^2 = e, \quad g_2^2 = e, \quad g_3^2 = e} \qquad \mathbf{(A)}$$

and consequently

$$\mathbf{g_i^{-1} = g_i} \quad \text{and} \quad \mathbf{g_i \circ g_k = g_k \circ g_i} \quad \text{for} \quad \mathbf{i, k = 1, 2, 3} \qquad \mathbf{(B)}$$

$$\mathbf{g_i \circ g_k = g_j} \quad \text{for} \quad \mathbf{i, j, k = 1, 2, 3.} \qquad \mathbf{(C)}$$

One among Many Powerful Group Properties

- One demonstrates that (A), (B) and (C) hold for all three groups

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◇ **Conclusion: We perform the mathematical analysis for one abstract group and apply the result for all homomorphic ones**

◇ **This may be a considerable intellectual gain - there are often many homomorphic images of the same 'generic' structure G**

Why Molecular Symmetries in Nuclear Physics?

Because of a 'new theorem' which says:

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◇ **Each symmetry-group of the mean-field Hamiltonian that is sufficiently rich in terms of the symmetry elements#) gives rise to the shell-closures analogous to the ones associated with the spherical symmetry at the proton and/or neutron particle numbers**

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#) Roughly speaking: ‘Rich in symmetry’ refers to the groups with as many irreducible representations as possible and the irreducible representation with as high a dimension as possible

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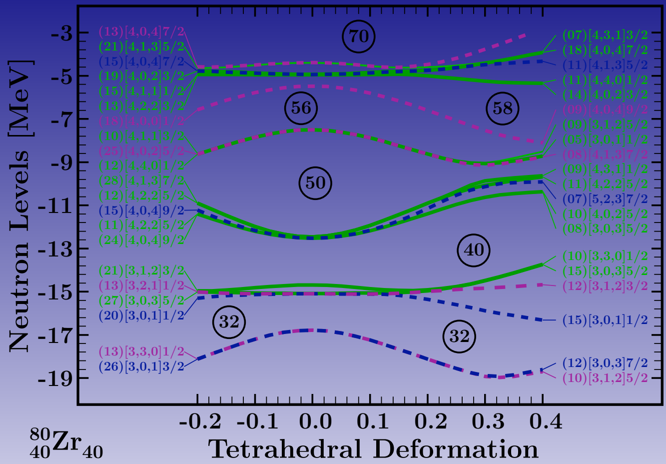
◇ **For instance in the case of tetrahedral symmetry (symmetry of a pyramid) the strong non-spherical shell-gaps are predicted for the proton and/or neutron particle numbers**

$$\boxed{32,\ 40,\ 56,\ 64,\ 70,\ 90,\ 132,\ \ldots}$$



Realistic single-particle neutron spectra in function of tetrahedral deformation. Observe the 'tetrahedral' gap at N=40 comparable in size to the gap at N=50.

Examples of Realistic Calculations for Nuclei

Realistic Nuclear Structure Calculations
Illustration for Selected Symmetries

[High-Spins: D_{2d}, D_{3d} and C_{3h}; Low-Spins T_d]

Nuclear D_{2d}-Group: 3D Examples

The nuclear D_{2d}-symmetric shapes have been predicted to coexist with the axial super-deformed shapes at high spins (JD and X. Li)

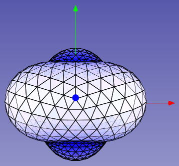


Figure: *Elongation axis*

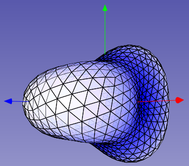


Figure: *Perspective 1*

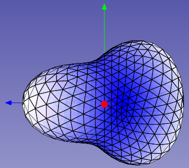


Figure: *Perspective 2*

Observations:

- Nuclear elongation in the range of $\alpha_{20} \sim (0.45 \rightarrow 0.55)$;
- Barriers between the coexisting minima $\sim (1 \rightarrow 2)$ MeV

Nuclear D_{3d}-Group: 3D Examples

The nuclear D_{3d}-symmetric shapes are expected at high spins; they correspond to superposition of α_{20} and α_{43} (inversion symmetric)

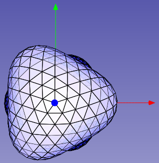
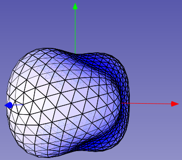


Figure: *Elongation axis*



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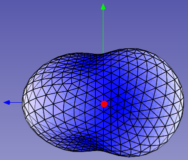


Figure: *Perspective 2*

Observations:

- Moderately elongated nuclei can form D_{3d}-symmetry shapes
- Probably seen already (remain mis-interpreted as tri-axiality)

Nuclear C_{3h}-Group ('Octupole'): 3D Examples

The nuclear C_{3h}-symmetric shapes are expected at high spins; they correspond to superposition of α_{20} and α_{33}

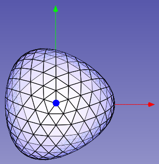
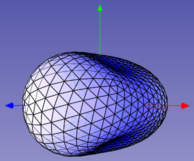


Figure: *Elongation axis*



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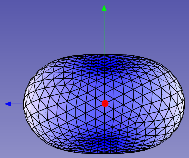


Figure: *Perspective 2*

Observations:

- Nuclei with C_{3h}-symmetry predicted to coexist with octupoles
- Probably seen already (and mis-interpreted in terms of $I^{\pi}=3^{-}$)

Nuclear Tetrahedral Shapes - 3D Examples

Illustrations below show the tetrahedral-symmetric surfaces at three increasing values of rank $\lambda = 3$ deformations t_1: 0.1, 0.2 and 0.3:

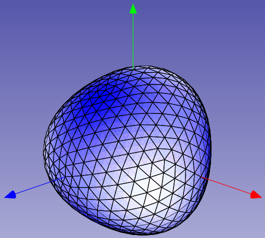


Figure: $t_3 = 0.1$

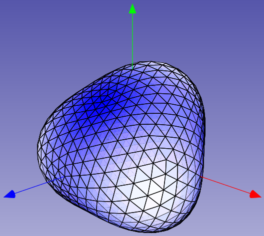


Figure: $t_3 = 0.2$

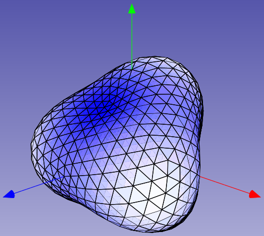


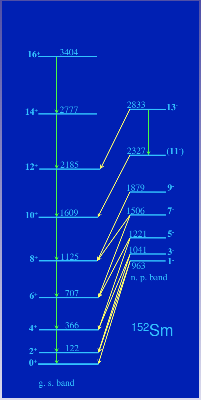
Figure: $t_3 = 0.3$

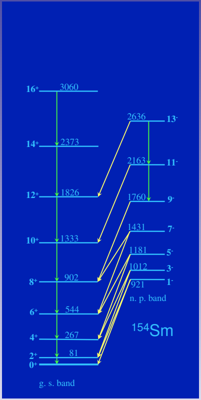
Observations:

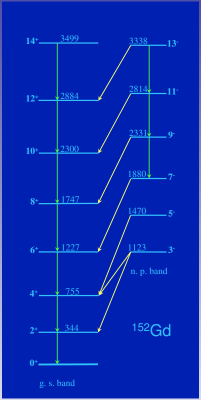
- There are infinitely many tetrahedral-symmetric surfaces
- Nuclear 'pyramids' do not resemble pyramids!

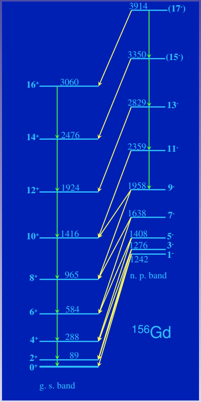
Possible Experimental Manifestation: No Q_2-Moments

Despite numerous tries nobody has ever succeed in observing E2's









The bands are identified thanks to the E1 transitions to the GSBs

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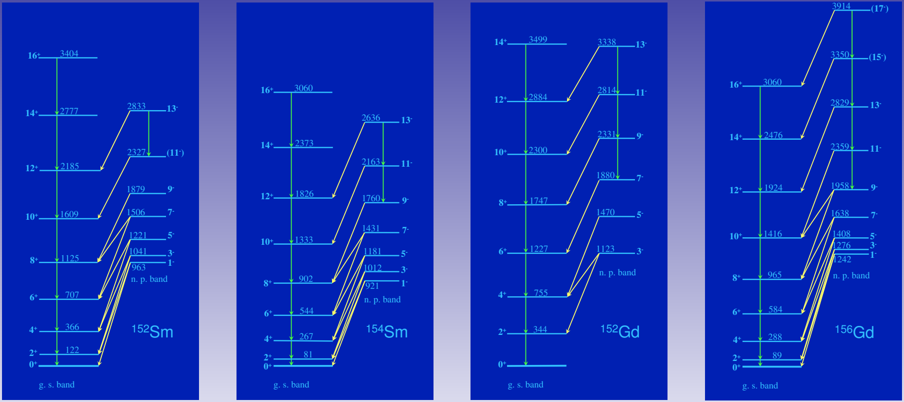

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Part II

Symmetry Groups and Quantum Mechanics [A Short Lesson to Refresh Your Memory]

Symmetry as Invariance of the Hamiltonian

- Symmetry of the system is equivalent to the invariance of $\hat{H}$:

$$\boxed{\hat{H}\,\psi_n = E_n\,\psi_n} \rightarrow \underbrace{[\mathcal{R}(\vec{\omega})\,\hat{H}\,\mathcal{R}^{-1}(\vec{\omega})]}_{\hat{H}'}\underbrace{[\mathcal{R}(\vec{\omega})\psi_n]}_{\psi'_n} = E_n\,\underbrace{[\mathcal{R}(\vec{\omega})\psi_n]}_{\psi'_n}$$

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$$\boxed{\hat{H}' = \hat{\mathcal{R}}(\vec{\omega})\hat{H}\,\hat{\mathcal{R}}^{-1}(\vec{\omega}) \quad \text{and} \quad \psi_n' = \hat{\mathcal{R}}(\vec{\omega})\psi_n}$$

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- Invariance of the Hamiltonian <u>does not mean</u> invariance of ψ_n
- Symmetry of the system implies an infinity of relations

$$\boxed{[\hat{\mathcal{R}}(\vec{\omega}), \hat{H}] = 0 \quad \forall\ \vec{\omega} \in \Omega \quad \leftrightarrow \quad \hat{\mathcal{R}} \in G}$$

Symmetry Groups - An Elementary! Illustration

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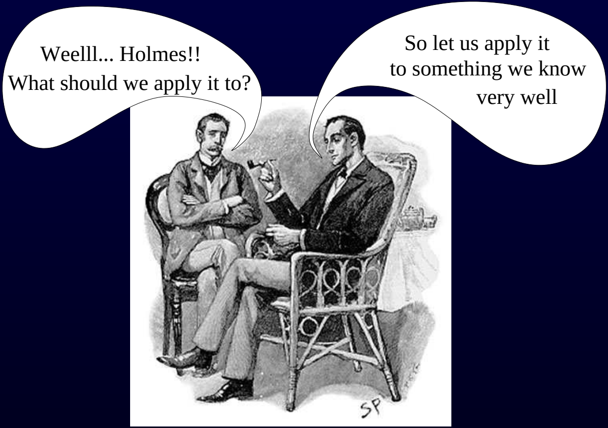

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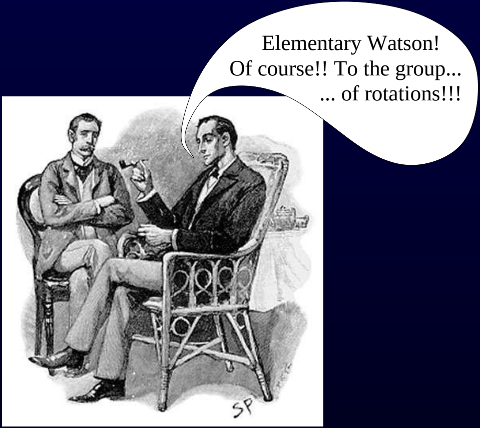

Expressing Rotations - Good Old Days at School

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- These are the elementary matrices that we have seen many times at school

$$\mathcal{O}_z : \quad \vec{r}\,' = \mathcal{R}_z(\omega_z)\,\vec{r} \quad \leftrightarrow \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\omega_z & \sin\omega_z & 0 \\ -\sin\omega_z & \cos\omega_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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$$\mathcal{O}_x : \quad \vec{r}\,' = \mathcal{R}_x(\omega_x)\,\vec{r} \quad \leftrightarrow \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_x & \sin\omega_x \\ 0 & -\sin\omega_x & \cos\omega_x \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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- There are infinitely many rotations $\leftrightarrow$ infinitely many ω_x, ω_y, $\omega_z \equiv \{\Omega\}$

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- To simplify the presentation we will consider one-dimensional rotations first
- These are the elementary matrices that we have seen many times at school

$$\mathcal{O}_z : \quad \vec{r}\,' = \mathcal{R}_z(\omega_z)\,\vec{r} \quad \leftrightarrow \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\omega_z & \sin\omega_z & 0 \\ -\sin\omega_z & \cos\omega_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathcal{O}_y : \quad \vec{r}\,' = \mathcal{R}_y(\omega_y)\,\vec{r} \quad \leftrightarrow \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\omega_y & 0 & -\sin\omega_y \\ 0 & 1 & 0 \\ \sin\omega_y & 0 & \cos\omega_y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathcal{O}_x : \quad \vec{r}\,' = \mathcal{R}_x(\omega_x)\,\vec{r} \quad \leftrightarrow \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_x & \sin\omega_x \\ 0 & -\sin\omega_x & \cos\omega_x \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- There are infinitely many rotations $\leftrightarrow$ infinitely many $\omega_x,\ \omega_y,\ \omega_z \equiv \{\Omega\}$
- Combining gives three-dimensional ones $\boxed{\mathcal{R}_x(\omega_x)\,\mathcal{R}_y(\omega_y)\,\mathcal{R}_z(\omega_z) \equiv \mathcal{R}(\vec{\omega}\,)}$

Rotation Groups and Infinitesimal Transformations

- Consider infinitesimal transformations with angles of rotation

$$\omega_x \longrightarrow \delta\omega_x, \quad \omega_y \longrightarrow \delta\omega_y \ \text{ and } \ \omega_z \longrightarrow \delta\omega_z$$

- At the limit of small angles, e.g.: ω_x:

$$\boxed{\cos\delta\omega_x \approx 1 \ \text{ and } \ \sin\delta\omega_x \approx \delta\omega_x}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta\omega_x & \sin\delta\omega_x \\ 0 & -\sin\delta\omega_x & \cos\delta\omega_x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta\omega_x \\ 0 & -\delta\omega_x & 1 \end{pmatrix} = \mathbb{1} + \frac{1}{i}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}\delta\omega_x$$

Rotation Groups and Infinitesimal Transformations

- **For infinitesimal rotations we obtain linearized expressions:**

Rotation Groups and Infinitesimal Transformations

- **For infinitesimal rotations we obtain linearized expressions:**

$$\hat{\mathcal{R}}_x(\delta\omega_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta\omega_x \\ 0 & -\delta\omega_x & 1 \end{pmatrix} = \mathbb{1} + \frac{1}{i}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \delta\omega_x \equiv \mathbb{1} - i\,\hat{g}_x\,\delta\omega_x,$$

$$\hat{\mathcal{R}}_y(\delta\omega_y) = \begin{pmatrix} 1 & 0 & -\delta\omega_y \\ 0 & 1 & 0 \\ \delta\omega_y & 0 & 1 \end{pmatrix} = \mathbb{1} + \frac{1}{i}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & i \\ i & 0 & 0 \end{pmatrix} \delta\omega_y \equiv \mathbb{1} - i\,\hat{g}_y\,\delta\omega_y,$$

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Infinity of Rotations with Just a Few Generators

- Above we have introduced new operators defined by

$$\hat{g}_x \overset{df.}{=} \frac{1}{i} \lim_{\omega_x \to 0} \frac{d\hat{\mathcal{R}}_x}{d\omega_x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

and

$$\hat{g}_y \overset{df.}{=} \frac{1}{i} \lim_{\omega_y \to 0} \frac{d\hat{\mathcal{R}}_y}{d\omega_y} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

as well as

$$\hat{g}_z \overset{df.}{=} \frac{1}{i} \lim_{\omega_z \to 0} \frac{d\hat{\mathcal{R}}_z}{d\omega_z} = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Generators are *hermitian operators* obeying the usual commutation relations

$$\boxed{\hat{g}^\dagger = \hat{g}} \quad \leftrightarrow \quad \boxed{[\hat{g}_x, \hat{g}_y] = i\hat{g}_z, \quad [\hat{g}_y, \hat{g}_z] = i\hat{g}_x, \quad [\hat{g}_z, \hat{g}_x] = i\hat{g}_y}$$

- The infinitesimal rotations can be expressed using one operator each:

$$\boxed{\hat{\mathcal{R}}_x(\delta\omega_x) = \mathbb{1} - i\hat{g}_x\,\delta\omega_x, \quad \hat{\mathcal{R}}_y(\delta\omega_y) = \mathbb{1} - i\hat{g}_y\,\delta\omega_y \;\text{ and }\; \hat{\mathcal{R}}_z(\delta\omega_z) = \mathbb{1} - i\hat{g}_z\,\delta\omega_z}$$

Properties of Generators of Rotations

- Introduce an oriented infinitesimal angle of a rotation about an $\vec{n}$ axis

$$\boxed{\boldsymbol{\delta\vec{\varphi} \equiv \vec{n}\,\delta\varphi \quad \leftrightarrow \quad \delta\omega_x \equiv n_x\delta\varphi, \quad \delta\omega_y \equiv n_y\delta\varphi \;\; \text{and} \;\; \delta\omega_z \equiv n_z\delta\varphi}}$$

- Since $\vec{n}^2 = 1$: $\Rightarrow \{n_x, n_y, n_z, \delta\varphi\}$ contains 3 and not 4 degrees of freedom
- Within the first order in terms of the infinitesimal angles of rotation

$$\boxed{\boldsymbol{\hat{\mathcal{R}}_x(\delta\omega_x)\,\hat{\mathcal{R}}_y(\delta\omega_y)\,\hat{\mathcal{R}}_z(\delta\omega_z) = \mathbb{1} - i\,(\hat{g}\cdot\vec{n})\,\delta\varphi \equiv \hat{\mathcal{R}}_{\vec{n}}(\delta\omega_x, \delta\omega_y, \delta\omega_z)}}$$

- Consider a finite angle φ and define $\Delta\varphi_N \equiv \varphi/N$. Since: $\lim\limits_{N\to\infty}(1\pm\dfrac{x}{N})^N = e^{\pm x}$

$$\hat{\mathcal{R}}_{\vec{n}}(\varphi) = [\hat{\mathcal{R}}_{\vec{n}}(\Delta\varphi_N)]^N = (1 - i\,\hat{g}\cdot\vec{n}\,\Delta\varphi_N)^N = (1 - i\,\hat{g}\cdot\vec{n}\,\varphi/N)^N \to \exp\left(-i\,\hat{g}\cdot\vec{n}\,\varphi\right)$$

Mathematical Generators and Physical Observables

- We demonstrated that a finite rotation about $\mathcal{O}_{\vec{n}}$ through φ is

$$\boxed{\hat{\mathcal{R}}_{\vec{\mathbf{n}}}(\varphi) = \mathbf{exp}\Big[-\mathbf{i}\,(\hat{\mathbf{g}}\cdot\vec{\mathbf{n}})\,\varphi \Big]}$$

This implies that the operator $\hat{g} \equiv \{\hat{g}_x, \hat{g}_y, \hat{g}_z\}$ is a generator of *both* the infinitesimal and finite rotations.

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This implies that the operator $\hat{g} \equiv \{\hat{g}_x, \hat{g}_y, \hat{g}_z\}$ is a generator of *both* the infinitesimal and finite rotations.

- Observe that generators $\{\hat{g}_\alpha\}$ and angular-momentum operators in quantum mechanics, apart from the Planck constant are identical

$$\boxed{[\hat{\mathbf{g}}_\mathbf{x}, \hat{\mathbf{g}}_\mathbf{y}] = \mathbf{i}\hat{\mathbf{g}}_\mathbf{z} \quad \leftrightarrow \quad [\hat{\ell}_\mathbf{x}, \hat{\ell}_\mathbf{y}] = \mathbf{i}\hbar\hat{\ell}_\mathbf{z}}$$

Operators $\hat{g}_\alpha$ and $\hat{\ell}_j$ can be connected by a unitary transformation; they differ merely by a choice of the reference frame & are equivalent

Observables and Infinitesimal Transformations

- Finite rotations can be written down equivalently as:

$$\hat{\mathcal{R}}_{\vec{n}}(\vec{\omega}) = \exp\left[-i\,(\hat{\ell}\cdot\vec{\omega})/\hbar\right] \quad \text{with} \;\; \vec{\omega} \overset{df}{=} \left(n_x\,\varphi, n_y\,\varphi, n_z\,\varphi\right)$$

- In particular for the infinitesimal angles $\delta\omega_x$, $\delta\omega_y$ and $\delta\omega_z$

$$\hat{\mathcal{R}}_x(\delta\omega_x) = \exp\left(-i\,\hat{\ell}_x\,\delta\omega_x/\hbar\right) \approx \mathbb{1} - i\,\hat{\ell}_x\,\delta\omega_x/\hbar$$
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$$\hat{\mathcal{R}}_z(\delta\omega_z) = \exp\left(-i\,\hat{\ell}_z\,\delta\omega_z/\hbar\right) \approx \mathbb{1} - i\,\hat{\ell}_z\,\delta\omega_z/\hbar\,.$$

- From invariance expression then follows that

$$0 = [\hat{\mathcal{R}}_x(\delta\omega_x), \hat{H}] = [\hat{\ell}_x, \hat{H}]\; i\delta\omega_x/\hbar, \quad \forall\,\delta\omega_x \quad \rightarrow \quad \boxed{[\hat{\ell}_x, \hat{H}] = 0}$$

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What Did We Learn: A Summary

Observables and Symmetries: Conclusions

Conclusion 1. Angular momentum operators can be identified with generators of all rotations in the 3-dimensional space $\{\hat{\ell}_\alpha\} \leftrightarrow \{\hat{g}_\alpha\}$

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Conclusion 2. If a physical system is spherically-symmetric its Hamiltonian must commute with all the three components of the angular momentum operator:

$$\boxed{[\hat{H}, \hat{\ell}_x] = 0, \quad [\hat{H}, \hat{\ell}_y] = 0 \quad \text{and} \quad [\hat{H}, \hat{\ell}_z] = 0.}$$

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Conclusion 3. Except for matrices defining the rotations, nowhere else we have used the properties of rotations $\longrightarrow$ our considerations generalise to other groups $\longrightarrow\longrightarrow\longrightarrow$ Namely $\longrightarrow\longrightarrow\longrightarrow$

Conservation Laws as the Result of Symmetries

- Time evolution of an observable $\hat{\mathcal{O}}$ in Heisenberg representation

$$i\hbar\frac{d\hat{\mathcal{O}}}{dt} = [\hat{\mathcal{O}}, \hat{H}] + i\hbar\frac{\partial\hat{\mathcal{O}}}{\partial t}$$

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$$\boxed{i\hbar\frac{d\hat{\ell}^2}{dt}=[\hat{\ell}^2,\hat{H}]=0 \quad\rightarrow\quad \ell^2=\text{constant of motion},}$$

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Conclusion : Eigen-values of $\hat{\ell}^2$ and $\hat{\ell}_z$ are constants of motion as the consequence of the spherical symmetry of the system

Generalisation for an Arbitrary Group of Symmetry

- Given group $G \equiv \{g\}$ with generators $\{\hat{g}_\alpha\}$ and parameters ω_α:

$$\mathbf{G} \ni \mathbf{g} = \exp[-\mathrm{i} \sum_{\boldsymbol{\alpha}} \hat{\mathbf{g}}_{\boldsymbol{\alpha}} \, \omega_{\boldsymbol{\alpha}}] \;\; \text{for} \;\; \boldsymbol{\alpha} = 1, 2, \, \ldots$$

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- Group G is a symmetry group of a physical system if there exists a representation in which

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- Hermitian $\{\hat{g}_\alpha\}$ will often be directly interpretable as observables. If in addition they do not depend explicitly on time, it follows that:

$$\boxed{\mathbf{i}\hbar \frac{\mathbf{d}\hat{\mathbf{g}}_\alpha}{\mathbf{dt}} = [\hat{\mathbf{g}}_\alpha, \hat{\mathbf{H}}] = 0 \quad \to \quad \hat{\mathbf{g}}_\alpha = \text{constant of motion}}$$

Part III

Fermion, Bosons, Spin and Isospin [Isospin Related Symmetries]

Identical Particles and the Permutation Group

- We consider a many-body system composed of n identical nucleons

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• We use position, linear momentum and spin, $\hat{r}$, $\hat{p}$, $\hat{s}$, to describe a particle $\hat{x} \equiv \{\hat{r}, \hat{p}, \hat{s}\}$. The Hamiltonian $\hat{H} = \hat{H}(\hat{x}_1, \hat{x}_2, \; \dots \; \hat{x}_n)$ must be symmetric under any permutation of particles 'i' and 'j' $\longrightarrow$

$$\begin{aligned} \hat{\mathcal{P}}_{\mathbf{ij}} \, \hat{\mathbf{H}}(\hat{\mathbf{x}}_1 \; \dots \; \hat{\mathbf{x}}_\mathbf{i} \; \dots \; \hat{\mathbf{x}}_\mathbf{j} \; \dots \; \hat{\mathbf{x}}_\mathbf{n}) \, \hat{\mathcal{P}}_{\mathbf{ij}}^{-1} \; &\overset{\mathrm{df}}{=} \; \hat{\mathbf{H}}(\hat{\mathbf{x}}_1 \; \dots \; \hat{\mathbf{x}}_\mathbf{j} \; \dots \; \hat{\mathbf{x}}_\mathbf{i} \; \dots \; \hat{\mathbf{x}}_\mathbf{n}) \\ &= \; \hat{\mathbf{H}}(\hat{\mathbf{x}}_1 \; \dots \; \hat{\mathbf{x}}_\mathbf{i} \; \dots \; \hat{\mathbf{x}}_\mathbf{j} \; \dots \; \hat{\mathbf{x}}_\mathbf{n}) \end{aligned}$$

and it follows: $\boxed{\hat{\mathcal{P}}_{\mathbf{ij}} \, \hat{\mathbf{H}} \, \hat{\mathcal{P}}_{\mathbf{ij}}^{-1} = \hat{\mathbf{H}} \;\; \longrightarrow \;\; [\hat{\mathcal{P}}_{\mathbf{ij}}, \hat{\mathbf{H}}] = 0, \quad \forall \; \mathbf{i} \neq \mathbf{j} \leq \mathbf{n}}$

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• Conclusions: 1. Both observables $\hat{\mathcal{P}}_{ij}$ and $\hat{H}$ can be diagonalized simultaneously; 2. The eigenvalues of $\hat{\mathcal{P}}_{ij}$ are constants of motion

3D Identical Particles: Either Fermions or Bosons

- Since $\hat{\mathcal{P}}_{ij}^2 = 1$ it follows that in $\boxed{\hat{\mathcal{P}}_{\mathbf{ij}}\boldsymbol{\Psi} = \mathbf{p}_{\mathbf{ij}}\boldsymbol{\Psi},}$ we must have

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- This implies that identical particles are either

$$\boxed{\textbf{Fermions :} \quad \hat{\mathcal{P}}_{\mathbf{ij}}\,\boldsymbol{\Psi}_{\mathbf{n_1},\,\dots\,\mathbf{n_i},\,\dots\mathbf{n_j},\,\dots\,\mathbf{n_n}} = -\boldsymbol{\Psi}_{\mathbf{n_1},\,\dots\,\mathbf{n_i},\,\dots\mathbf{n_j},\,\dots\,\mathbf{n_n}},\ \forall\ \mathbf{i},\mathbf{j}}$$

or

$$\boxed{\textbf{Bosons :} \quad \hat{\mathcal{P}}_{\mathbf{ij}}\,\boldsymbol{\Phi}_{\mathbf{n_1},\,\dots\,\mathbf{n_i},\,\dots\mathbf{n_j},\,\dots\,\mathbf{n_n}} = +\boldsymbol{\Phi}_{\mathbf{n_1},\,\dots\,\mathbf{n_i},\,\dots\mathbf{n_j},\,\dots\,\mathbf{n_n}},\ \forall\ \mathbf{i},\mathbf{j}.}$$

and this $\longrightarrow$ all life-long for all identical particles of a given type

About Identical Particles: Pauli Principle

- We say that the wave-functions for identical Fermions are totally anti-symmetric and those for Bosons are totally symmetric

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Physical many-body states are either totally symmetric or totally anti-symmetric

Anti-Symmetrising Fermion Wave-Functions

Fermion Wave-Functions: Anti-Symmetrisation

• Let us begin by posing a certain elementary problem that many of you know already how to tackle:

What is the structure of $s = \frac{1}{2}$ two-particle wave functions at $\vec{r}_1$ and $\vec{r}_2$?

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What is the structure of $s = \frac{1}{2}$ two-particle wave functions at $\vec{r}_1$ and $\vec{r}_2$?

• The wave functions depend on the spatial parts $\varphi_\alpha(\vec{r}\,)$ and $\varphi_\beta(\vec{r}\,)$ and the spin part χ_{s,s_z}: the total wave functions must be antisymmetric:

$$\boxed{\boldsymbol{\Psi}_{\alpha\beta} \sim \textbf{Anti-symm.}[\varphi_\alpha(\vec{\mathbf{r}}_1\,), \varphi_\beta(\vec{\mathbf{r}}_2\,)] \times \textbf{Symm.}[\chi_{\mathrm{s},\mathrm{s}_{z,1}}, \chi_{\mathrm{s},\mathrm{s}_{z,2}}]}$$

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Space Symmetrisation and Anti-Symmetrisation

• The symmetrisation and anti-symmetrisation in space can be done in a simple, unique manner: for spatially anti-symmetric 2-nucleon functions

$$\begin{aligned}\hat{\mathcal{A}}\Psi_{\alpha\beta}(\vec{r}_1,\vec{r}_2) &= \frac{1}{\sqrt{2}}[\varphi_\alpha(\vec{r}_1)\cdot\varphi_\beta(\vec{r}_2)-\varphi_\alpha(\vec{r}_2)\cdot\varphi_\beta(\vec{r}_1)]\\ &\leftrightarrow \frac{1}{\sqrt{2}}[\varphi_\alpha(\vec{r}_1)\cdot\varphi_\beta(\vec{r}_2)-\varphi_\beta(\vec{r}_1)\cdot\varphi_\alpha(\vec{r}_2)]\end{aligned}$$

Space Symmetrisation and Anti-Symmetrisation

• The symmetrisation and anti-symmetrisation in space can be done in a simple, unique manner: for spatially anti-symmetric 2-nucleon functions

$$\begin{aligned} \hat{\mathcal{A}}\Psi_{\alpha\beta}(\vec{r}_1,\vec{r}_2) &= \frac{1}{\sqrt{2}}[\varphi_\alpha(\vec{r}_1)\cdot\varphi_\beta(\vec{r}_2)-\varphi_\alpha(\vec{r}_2)\cdot\varphi_\beta(\vec{r}_1)] \\ &\leftrightarrow \frac{1}{\sqrt{2}}[\varphi_\alpha(\vec{r}_1)\cdot\varphi_\beta(\vec{r}_2)-\varphi_\beta(\vec{r}_1)\cdot\varphi_\alpha(\vec{r}_2)] \end{aligned}$$

and for spatially-symmetric 2-nucleon functions we find:

$$\begin{aligned} \hat{\mathcal{S}}\Psi_{\alpha\beta}(\vec{r}_1,\vec{r}_2) &= \frac{1}{\sqrt{2}}[\varphi_\alpha(\vec{r}_1)\cdot\varphi_\beta(\vec{r}_2)+\varphi_\alpha(\vec{r}_2)\cdot\varphi_\beta(\vec{r}_1)] \\ &\leftrightarrow \frac{1}{\sqrt{2}}[\varphi_\alpha(\vec{r}_1)\cdot\varphi_\beta(\vec{r}_2)+\varphi_\beta(\vec{r}_1)\cdot\varphi_\alpha(\vec{r}_2)] \end{aligned}$$

Fermion Spin-Symmetrisation & Anti-Symmetrisation

- We have <u>three</u> spin-symmetric wave-functions for $s_1 = s_2 = \frac{1}{2}$, namely:

$$\boxed{\chi^{1,2}_{S=1,S_z=+1} = \chi^{(1)}_{\frac{1}{2},+\frac{1}{2}}\chi^{(2)}_{\frac{1}{2},+\frac{1}{2}} \quad \text{and} \quad \chi^{1,2}_{S=1,S_z=-1} = \chi^{(1)}_{\frac{1}{2},-\frac{1}{2}}\chi^{(2)}_{\frac{1}{2},-\frac{1}{2}}}$$

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$$\boxed{\chi^{1,2}_{S=1,S_z=0} = \frac{1}{\sqrt{2}}\left[\chi^{(1)}_{\frac{1}{2},+\frac{1}{2}}\chi^{(2)}_{\frac{1}{2},-\frac{1}{2}} + \chi^{(1)}_{\frac{1}{2},-\frac{1}{2}}\chi^{(2)}_{\frac{1}{2},+\frac{1}{2}}\right]}$$

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- We have only one spin-anti-symmetric function for the two nucleons

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- These are spin-triplet $\chi^{1,2}_{S=1,S_z}$ and spin-singlet $\chi^{1,2}_{S=0,S_z=0}$ wave-functions

Isospin - The Younger Brother of Spin

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• We say that the nucleon has isospin $t = \frac{1}{2}$ and $t_z = -\frac{1}{2}$ if the charge is $q = +1$e (i.e. proton) while $t_z = +\frac{1}{2}$ if the charge is $q = 0$ (i.e. neutron).

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• Analogy:

$$\boxed{\chi_{\mathbf{s}=\frac{1}{2},\mathbf{s_z}=\pm\frac{1}{2}} \leftrightarrow \quad \text{spin-up vs. spin-down}}$$

$$\boxed{\chi_{\mathbf{t}=\frac{1}{2},\mathbf{t_z}=\pm\frac{1}{2}} \leftrightarrow \text{charge-on vs. charge-off}}$$

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- These are: iso-vector $\chi^{1,2}_{T=1,T_z}$, and iso-scalar, $\chi^{1,2}_{T=0,T_z=0}$ wave functions

Pauli Principle Generalized for the Nucleons

- Generalized Pauli principle implies total anti-symmetry of wave-functions

$$\boxed{\boldsymbol{\Psi}_{\alpha\beta} = \psi_{\alpha\beta}(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)\; \chi^{\alpha,\beta}_{\mathsf{S},\mathsf{S}_z}\; \chi^{\alpha,\beta}_{\mathsf{T},\mathsf{T}_z} \;\leftrightarrow\; \text{anti-symmetric}}$$

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- The physically acceptable two-body wave functions are

$$\psi^{\mathcal{A}}_{12}\,\chi^{\mathcal{S}}_{S,S_z}\,\chi^{\mathcal{S}}_{T,T_z} \;\longrightarrow\; \psi^{\mathcal{A}}_{12}\,\chi^{\mathcal{S}}_{S,S_z}\,[\underbrace{\chi^{\mathcal{S}}_{T=1,-1}}_{pp} \text{ or } \underbrace{\chi^{\mathcal{S}}_{T=1,0}}_{pn} \text{ or } \underbrace{\chi^{\mathcal{S}}_{T=1,+1}}_{nn}]$$

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EJC Morning Quiz

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What can be said about the dominating feature of the nucleon-nucleon interactions: More attractive - or just the opposite?

Isospin-Related Symmetries

A Short Reminder

Isospin - a Younger Brother of Spin: Formulae

- A comment about various forms of notation for states (wave f.):

$$\begin{pmatrix}0\\1\end{pmatrix} \leftrightarrow \chi_p \leftrightarrow \chi_{\frac{1}{2},-\frac{1}{2}} \xleftarrow{\;p\;} \boxed{\chi_{t,t_z}} \xrightarrow{\;n\;} \chi_{\frac{1}{2},+\frac{1}{2}} \leftrightarrow \chi_n \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$$

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$$\hat{t} \equiv \frac{1}{2}\,\hat{\tau} \equiv \frac{1}{2}\{\hat{\tau}_x, \hat{\tau}_y, \hat{\tau}_z\} \quad \leftrightarrow \quad \left\{\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\-i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right\}$$

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- Analogy with the spin operators

$$\hat{s} \equiv \tfrac{1}{2}\,\hat{\sigma} \equiv \tfrac{1}{2}\{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\} \quad \leftrightarrow \quad \left\{ \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\-i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \right\}$$

The Most General Dependence of Nuclear Forces...

... on Spin and/or Isospin

- Recall the elementary algebraic property of the 3 Pauli matrices:

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- The most general dependence of any potential on σ_α has the form

$$v(\sigma_\alpha) = \sum_{n=0}^{\infty} \frac{v^{(n)}|_0}{n!} \cdot \sigma_\alpha^n = \sum_{k=0}^{\infty} \left\{ \frac{v^{(2k)}|_0}{(2k)!} \cdot \underbrace{(\sigma_\alpha)^{2k}}_{\mathbb{1}} + \frac{v^{(2k+1)}|_0}{(2k+1)!} \cdot \underbrace{(\sigma_\alpha)^{2k+1}}_{\sigma_\alpha} \right\}$$

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- **Conclusion:** The most general dependence on spin or isospin is

$$\boxed{\mathbf{v}(\sigma_\alpha) = \mathbf{a} \cdot \mathbb{1} + \mathbf{b} \cdot \sigma_\alpha \;\rightarrow\; \mathbf{v}(\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2) = \mathbf{A}_0\, \mathbb{1} + \mathbf{B}_{\mathbf{s}\cdot\mathbf{s}}\, \hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2}$$

where **A** and **B** are in general function functions of operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$

A Question in Passing [Another Quiz?]

- As we just recalled, spin is a vector-operator $\hat{s} = \{\hat{s}_x, \hat{s}_y, \hat{s}_z\} \in 3D$
- When rotating the coordinate frame the spin transform like this:

$$\boxed{\begin{pmatrix} \hat{s}'_x \\ \hat{s}'_y \\ \hat{s}'_x \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} \hat{s}_x \\ \hat{s}_y \\ \hat{s}_x \end{pmatrix}}$$

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- The wave function for the spin $s = \frac{1}{2}$ particle has two components

$$\chi_{s=\frac{1}{2}} = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$$

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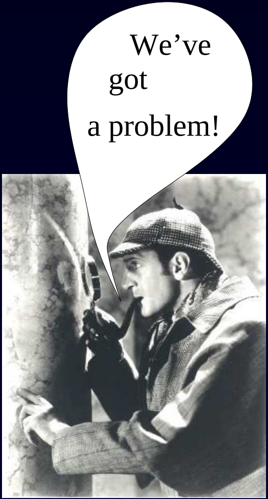
- **Did you think:** How does this wave function transform?

$$\chi'_{\mathbf{s}=\frac{1}{2}} = [\ ?\ ] \times \chi_{\mathbf{s}=\frac{1}{2}}$$


We've
got
a problem!


We've got a problem!
What kind of problem?

If I have a vector $\vec{r} \in \mathbb{R}^3$
and want to rotate it, I just go ahead.
Take a 3×3 matrix $\mathcal{R}(\vec{\omega})$ ... aaaand:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \mathcal{R}_{13} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \mathcal{R}_{23} \\ \mathcal{R}_{31} & \mathcal{R}_{32} & \mathcal{R}_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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You always managed to surprize me ...

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... and ?

Take a particle from the shell j=13/2 ...
It has (2j+1) components rather than 3(!)

Do you know where to stick
your 3×3 matrix $\mathcal{R}$?

I don't dare guessing ...

Watson!

Cable immediately Professor Wigner.

Ask details about the theory of group representations!

Representations of the group of rotations!

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Representations of the group of rotations!

... But Holmes!
Wagner's preoccupation
is rather ... music!

Watson!

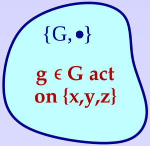
Stop playing on my nerves.

Take aspirin ... and vitamins ...
and wash your hears!

I said: Eugene WIGNER!

Q. Mechanics and Group Theory on One Page

Starting Point: A Group
[e.g. transformations]



Q. Mechanics and Group Theory on One Page

Starting Point: A Group
[e.g. transformations]

$\{G, \bullet\}$

$g \in G$ act on $\{x,y,z\}$

Vector Space
$\{V\} \Leftrightarrow \{\psi\}$
$\psi' = D(g)\,\psi$

Hamiltonian H acts on wave functions

Q. Mechanics and Group Theory on One Page

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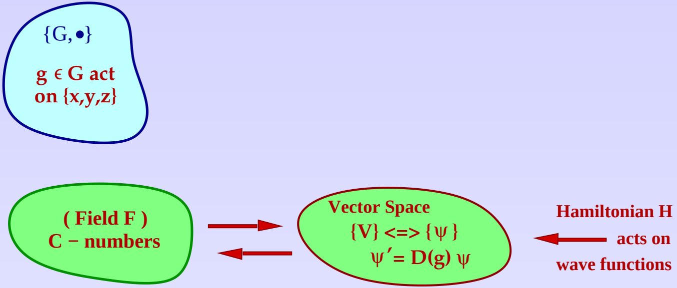

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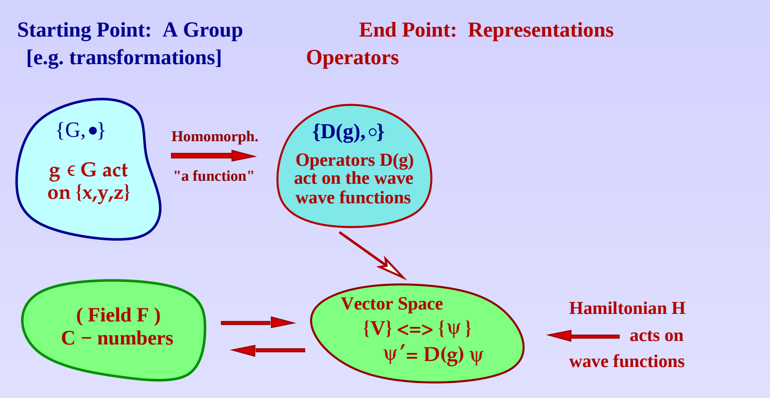
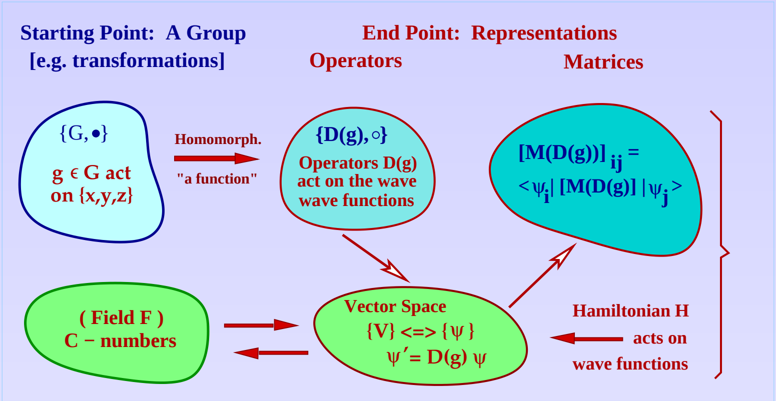


Q. Mechanics and Group Theory on One Page



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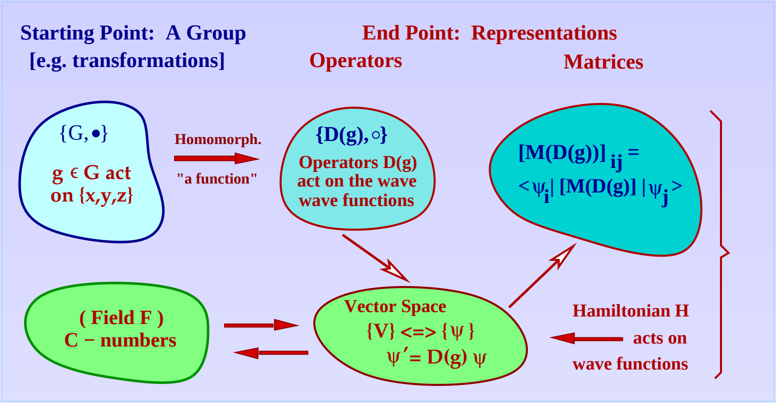

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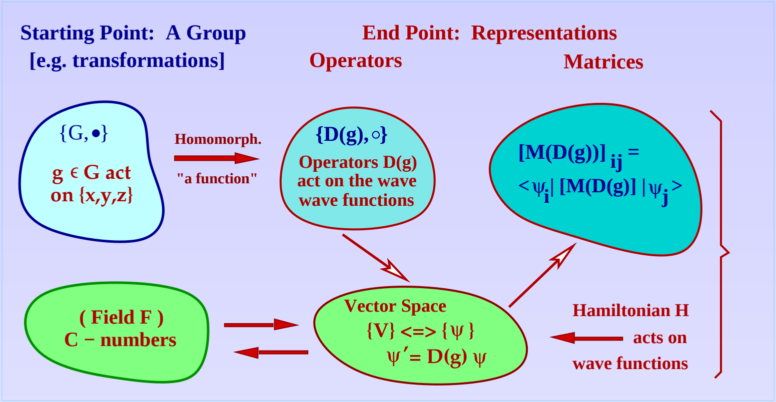

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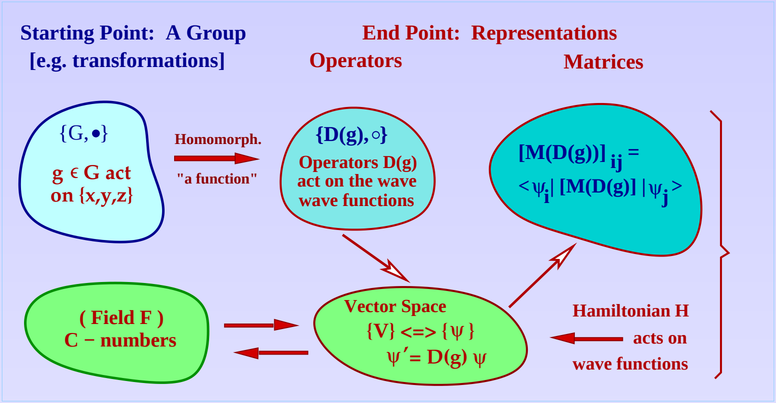

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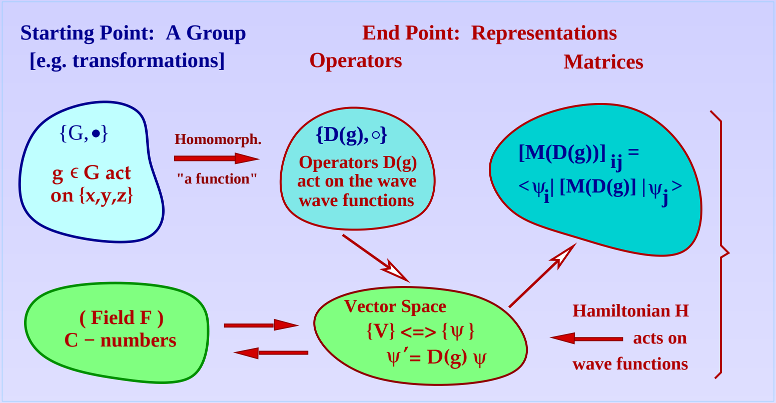

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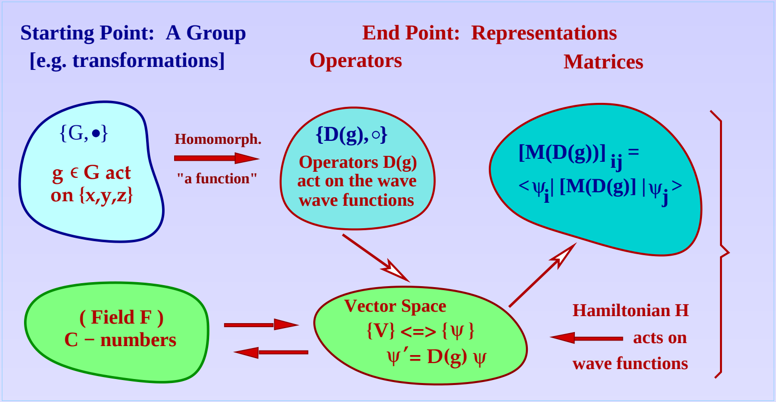

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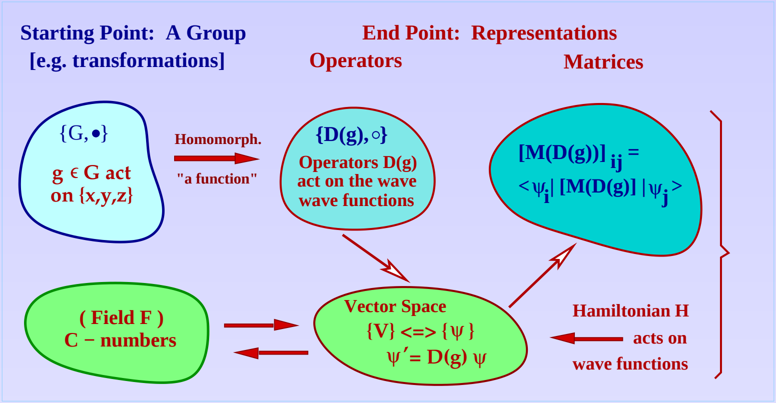

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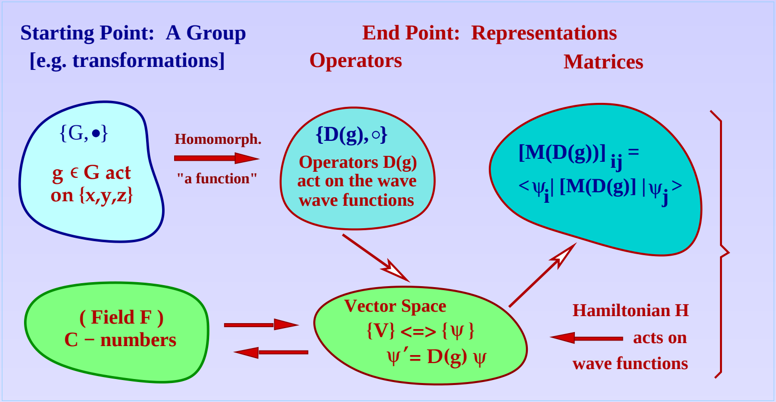

∘ Consider a group $\{G, \bullet\}$; ∘ Consider a vector space V over the field F; ∘ Introduce an ensemble of linear operators $\mathcal{D}$ that are functions of $g \in G$; ∘ They by definition, act in V. ∘ We assume they form a new group $\{\mathcal{D}(g), \circ\}$. ∘ If this group is homomorphic to G we call it a representation of G.

Formal Definition of a Group Representation

• Consider a group $\{G, \bullet\}$, a vector space V over the field F, and an ensemble of linear operators $\mathcal{D}$ that are functions of $g \in G$ and, by definition, act in V.

• We assume that those operators form a group $\{\mathcal{D}(g), \circ\}$. Suppose that there exists a homomorphic mapping between the two groups, $G \to \mathcal{D}(g)$ so that:

$$\mathcal{D}(g_1 \circ g_2) = \mathcal{D}(g_1) \cdot \mathcal{D}(g_2)$$

as well as

$$\mathcal{D}(g^{-1}) = \mathcal{D}^{-1}(g)$$

where from it follows that for the neutral elements

$$\mathcal{D}(e) = \mathbb{1}.$$

• When the above conditions are satisfied, the group composed of operators $\{\mathcal{D}(g), \cdot\}$ is called *representation of the original group G*.

After the intrusion from the British experts

BACK TO THE ISOSPIN

Charge and Isospin Operators

- For the neutron and proton wave functions we find the identities

$$\hat{t}_z\,\chi_n = +\tfrac{1}{2}\,\chi_n \;\;\longrightarrow\;\; \left(\tfrac{1}{2} - \hat{t}_z\right)\chi_n = 0 \cdot \chi_n$$

$$\hat{t}_z\,\chi_p = -\tfrac{1}{2}\,\chi_p \;\;\longrightarrow\;\; \left(\tfrac{1}{2} - \hat{t}_z\right)\chi_p = 1 \cdot \chi_p$$

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- The above relations suggest the definition of the charge operator:

$$\boxed{\hat{q} \equiv e\cdot\big(\tfrac{1}{2}-\hat{t}_z\big) \quad\rightarrow\quad \hat{q}\,\chi_p = 1\cdot e\,\chi_p \quad \text{and} \quad \hat{q}\,\chi_n = 0\cdot e\,\chi_n}$$

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- It follows the definition of the total nuclear charge operator:

$$\boxed{\hat{Q} \overset{df}{=} \textstyle\sum_{i=1}^{A} \hat{q}_i = \tfrac{1}{2}\,e \cdot A - e \sum_{i=1}^{A} \hat{t}_z(i) \equiv \tfrac{1}{2}\,e \cdot A - e \cdot \hat{T}_z}$$

Charge Conservation in Nature and the Isospin

- Experimental results are compatible with charge conservation

$$\boxed{[\hat{H}, \hat{Q}] = 0} \quad \rightarrow \quad [\hat{H}, \tfrac{1}{2}\, e \cdot A - e \cdot \hat{T}_z] = 0 \quad \rightarrow \quad \boxed{[\hat{H}, \hat{T}_z] = 0}$$

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◇ **This implies a constraint on the other two components**

$$[\hat{\mathbf{H}},(\hat{\mathbf{T}}_x^2+\hat{\mathbf{T}}_y^2)]=0$$

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◇ **What can be said about the other components alone?**

From Charge Conservation to Charge Independence

- Suppose, *as a working hypothesis*, that for all other components

$$\boxed{[\hat{H}, \hat{T}_x] = 0, \;\; [\hat{H}, \hat{T}_y] = 0 \;\text{ and }\; [\hat{H}, \hat{T}_z] = 0 \;\; \rightarrow \;\; [\hat{H}, \hat{T}^2] = 0}$$

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- When this happens the following standard four relations are valid

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- This allows introducing the common solutions to $\hat{H}$, $\hat{T}^2$ and $\hat{T}_z$:

$$\boxed{\hat{T}^2 \,|T, T_z\rangle = T(T+1)\,|T, T_z\rangle \;\text{ and }\; \hat{T}_z\,|T, T_z\rangle = T_z\,|T, T_z\rangle}$$

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- These relations allow formulating charge independence hypothesis

$\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow$

Formulating the Charge Independence Hypothesis

- The Eckart-Wigner theorem implies the projection-independence

$$\langle \mathbf{T}, \mathbf{T_z} | \hat{\mathbf{H}} | \mathbf{T}', \mathbf{T}'_z \rangle = \delta_{\mathbf{T}\mathbf{T}'}\, \delta_{\mathbf{T_z}\mathbf{T}'_z}\, \langle \mathbf{T} || \hat{\mathbf{H}} || \mathbf{T}' \rangle \leftarrow \text{red. m. element}$$

- In other words we have the following particular relations for **T=1**

$$\langle \mathbf{1}, +\mathbf{1} | \hat{\mathbf{H}} | \mathbf{1}, +\mathbf{1} \rangle = \langle \mathbf{1}, \mathbf{0} | \hat{\mathbf{H}} | \mathbf{1}, \mathbf{0} \rangle = \langle \mathbf{1}, -\mathbf{1} | \hat{\mathbf{H}} | \mathbf{1}, -\mathbf{1} \rangle$$

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$$\underbrace{\langle 1, +1 | \hat{\mathbf{H}} | 1, +1 \rangle}_{\mathbf{H}^{\mathbf{T=1}}_{\mathbf{n-n}}} = \underbrace{\langle 1, 0 | \hat{\mathbf{H}} | 1, 0 \rangle}_{\mathbf{H}^{\mathbf{T=1}}_{\mathbf{p-n}}} = \underbrace{\langle 1, -1 | \hat{\mathbf{H}} | 1, -1 \rangle}_{\mathbf{H}^{\mathbf{T=1}}_{\mathbf{p-p}}}$$

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- This is why $[\hat{H}, \hat{T}] = 0$ is called <u>Charge Independence Hypothesis</u>

- Observe that in general $\langle T || \hat{H} || T \rangle = f(T)$ i.e. a function of $T \to$

$$\boxed{\langle \mathbf{T} = \mathbf{1} || \hat{\mathbf{H}} || \mathbf{T} = \mathbf{1} \rangle \neq \langle \mathbf{T} = \mathbf{0} || \hat{\mathbf{H}} || \mathbf{T} = \mathbf{0} \rangle}$$

Charge Independence and the N-N Interaction

- Since $[\hat{H}, \hat{T}^2] = 0$, interaction term must be a functional of $\hat{T}^2$

$$\boxed{\Rightarrow \quad \text{To lowest order :} \quad V_{N-N} \sim \big(\alpha\, \mathbb{1} + \beta\, \hat{T}^2\big)}$$

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$$\hat{T}^2 = \left[\hat{t}^{(1)}\right]^2 + \left[\hat{t}^{(2)}\right]^2 + 2\,\hat{t}^{(1)} \cdot \hat{t}^{(2)} \quad \rightarrow \quad \hat{T}^2 \leftrightarrow \tfrac{3}{4} + \tfrac{3}{4} + 2\,\hat{t}^{(1)} \cdot \hat{t}^{(2)}$$

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- Charge-Independence implies the isospin dependence of interaction

$$\boxed{\mathbf{V_{N-N}} \sim \left(\mathbf{a}\, \mathbb{1} + \mathbf{b}\, \hat{\mathbf{t}}^{(1)} \cdot \hat{\mathbf{t}}^{(2)}\right)}$$

where **a** and **b** are arbitrary functions of nucleon observables $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$

Charge Symmetry Hypothesis

• Spectra of nuclear pairs such as $\left[{}^{15}_{7}\mathrm{N}_8 \leftrightarrow {}^{15}_{8}\mathrm{O}_7\right]$, $\left[{}^{17}_{8}\mathrm{O}_9 \leftrightarrow {}^{17}_{9}\mathrm{F}_8\right]$, $\left[{}^{39}_{19}\mathrm{K}_{20} \leftrightarrow {}^{39}_{20}\mathrm{Ca}_{19}\right]$, $\left[{}^{41}_{20}\mathrm{Ca}_{21} \leftrightarrow {}^{41}_{21}\mathrm{Sc}_{20}\right]$, $\left[{}^{42}_{20}\mathrm{Ca}_{22} \leftrightarrow {}^{42}_{22}\mathrm{Ti}_{20}\right]$, ... etc. shows very strong similarities and suggest that the rotation $\mathcal{R}^T_\perp(\pi)$

$$\boxed{(\mathbf{Z} \longrightarrow \mathbf{N},\ \mathbf{N} \longrightarrow \mathbf{Z}) \quad \leftrightarrow \quad \mathcal{R}^{\mathbf{T}}_\perp(\pi) \overset{\mathbf{df}}{=} \mathbf{exp}[\mathbf{i}\,\pi \hat{\mathbf{T}}_\perp]}$$

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- This approx. symmetry of interaction is called Charge Symmetry

Charge Symmetry Hypothesis: Consequences

- Since $\hat{t}^{(1)} \cdot \hat{t}^{(2)}$ is scalar and since $\hat{\mathcal{P}}_{CS}$ changes signs of $\hat{t}_z^{(1)}$ and $\hat{t}_z^{(2)}$

$$\boxed{\left[\hat{\mathcal{P}}_{\mathbf{CS}}, \hat{\mathbf{t}}^{(1)} \cdot \hat{\mathbf{t}}^{(2)}\right] = 0 \quad \textbf{and} \quad \left[\hat{\mathcal{P}}_{\mathbf{CS}}, \hat{\mathbf{t}}_{\mathbf{z}}^{(1)} \cdot \hat{\mathbf{t}}_{\mathbf{z}}^{(2)}\right] = 0 \qquad (*)}$$

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- It follows that the most general charge-symmetric Hamiltonian is

$$\boxed{\hat{\mathbf{H}} = \mathbf{A_0}\, \mathbb{1} + \mathbf{B_{t \cdot t}}\, \hat{\mathbf{t}}^{(1)} \cdot \hat{\mathbf{t}}^{(2)} + \mathbf{C_{t_z, t_z}}\, \hat{\mathbf{t}}_{\mathbf{z}}^{(1)}\, \hat{\mathbf{t}}_{\mathbf{z}}^{(2)}}$$

with $\mathbf{A_0}$, $\mathbf{B_{t \cdot t}}$ and $\mathbf{C_{t_z, t_z}}$ - arbitrary functions of observables $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$

Isospin-Structure of the Coulomb Interaction

- The two-nucleon Coulomb potential is generally written down as:

$$\boxed{\mathbf{V}_{\mathbf{C}}^{(1,2)} = \frac{\mathbf{e}^2}{|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2|} \times \left(\frac{1}{2} - \hat{\mathbf{t}}_z^{(1)}\right)\left(\frac{1}{2} - \hat{\mathbf{t}}_z^{(2)}\right)}$$

- It can be rewritten in terms of iso-scalar, iso-vector and iso-tensor

$$\mathbf{V}_{\mathbf{C}}^{(1,2)} = \frac{\mathbf{e}^2}{|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2|}\Big\{\Big[\tfrac{1}{4} + \tfrac{1}{3}\,\hat{\mathbf{t}}^{(1)} \cdot \hat{\mathbf{t}}^{(2)}\Big] - \tfrac{1}{2}\Big[\hat{\mathbf{t}}_z^{(1)} + \hat{\mathbf{t}}_z^{(2)}\Big] + \Big[\hat{\mathbf{t}}_z^{(1)}\,\hat{\mathbf{t}}_z^{(2)} - \tfrac{1}{3}\hat{\mathbf{t}}^{(1)} \cdot \hat{\mathbf{t}}^{(2)}\Big]\Big\}$$

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$$\boxed{V_C^{(1,2)} = \frac{e^2}{|\vec{r}_1 - \vec{r}_2|} \times \left(\frac{1}{2} - \hat{t}_z^{(1)}\right)\left(\frac{1}{2} - \hat{t}_z^{(2)}\right)}$$

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$$V_C^{(1,2)} = \frac{e^2}{|\vec{r}_1 - \vec{r}_2|}\Big\{ \underbrace{\left[\tfrac{1}{4} + \tfrac{1}{3}\,\hat{t}^{(1)} \cdot \hat{t}^{(2)}\right]}_{\text{iso-scalar: } V_{I-S}^{(1,2)}} - \tfrac{1}{2}\, \underbrace{\left[\hat{t}_z^{(1)} + \hat{t}_z^{(2)}\right]}_{\text{iso-vector: } V_{I-V}^{(1,2)}} + \underbrace{\left[\hat{t}_z^{(1)}\,\hat{t}_z^{(2)} - \tfrac{1}{3}\hat{t}^{(1)} \cdot \hat{t}^{(2)}\right]}_{\text{iso-tensor: } V_{I-T}^{(1,2)}} \Big\}$$

- Expressions below give rise to various symmetry-breaking effects

$$\boxed{\left[\hat{T}^2, V_{I-S}^{(1,2)}\right] = 0, \quad \left[\hat{T}^2, V_{I-V}^{(1,2)}\right] \neq 0, \quad \left[\hat{T}^2, V_{I-T}^{(1,2)}\right] \neq 0}$$

$$\boxed{\left[\hat{\hat{\mathcal{P}}}_{CS}, V_{I-S}^{(1,2)}\right] = 0, \quad \left[\hat{\hat{\mathcal{P}}}_{CS}, V_{I-V}^{(1,2)}\right] = 0, \quad \left[\hat{\hat{\mathcal{P}}}_{CS}, V_{I-T}^{(1,2)}\right] \neq 0}$$

Part IV

Two-Nucleon Interactions and Their Symmetries

Relativity and the Implied Symmetries

Relativity Principle and Space-Time Properties

- **According to Einstein's formulation* of the relativity principle:**

If a system of coordinates $\boldsymbol{\Sigma}$ is chosen so that, in relation to it, physical laws hold good in their simplest form, the same laws hold good in relation to any other system of coordinates $\boldsymbol{\Sigma'}$ moving in uniform translation relatively to $\boldsymbol{\Sigma}$.

Albert Einstein: "The foundation of the general theory of relativity"

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- **Uniformity of Space: All points in our 3D space are equivalent**
- **Isotropy of Space: All directions in our 3D space are equivalent**
- **Uniformity of Time: No time instant in our space is priviledged**

* In a non-relativistic approach as the one which follows we may use the historically earlier, Galilean formulation.

Hamiltonian Form Allowed by Symmetries (1)

Hermiticity of the Hamiltonian: *We must assume that the Hamiltonian is an observable and therefore Hermitian*

$$\boxed{\hat{H}(\hat{x}_1,\hat{x}_2) \equiv \hat{t}_1 + \hat{t}_2 + \hat{V}(\hat{x}_1,\hat{x}_2); \quad \hat{H}^\dagger = \hat{H} \;\rightarrow\; \hat{V}^\dagger = \hat{V}}$$

Nucleons Are Indistinguishable: *It follows that the Hamiltonian must be symmetric with respect to exchange of the two particles*

$$\boxed{\hat{H}(\hat{x}_1,\hat{x}_2) = \hat{H}(\hat{x}_2,\hat{x}_1) \;\rightarrow\; \hat{V}(\hat{x}_1,\hat{x}_2) = \hat{V}(\hat{x}_2,\hat{x}_1)}$$

Translational Invariance: *Reference frames are equivalent, Hamiltonians expressed in Σ - and Σ' related to Σ by translation - must be identical*

$$\boxed{\widehat{V} = \widehat{V}[(\hat{r}_1 - \hat{r}_2); (\hat{p}_1,\hat{p}_2); (\hat{s}_1,\hat{s}_2); (\hat{t}_1,\hat{t}_2)]}$$

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Hamiltonian Form Allowed by Symmetries (2)

Equivalence of Inertial Frames: *Consider a reference frame Σ' moving with respect to Σ with an arbitrary constant velocity $\vec{v}$*

$$\Sigma: \ \{\vec{v}_1, \vec{v}_2\} \quad \to \quad \Sigma': \ \left.\begin{array}{l} \vec{v}_1 \to \vec{v}_1{}' = \vec{v}_1 + \vec{v} \\ \vec{v}_2 \to \vec{v}_2{}' = \vec{v}_2 + \vec{v} \end{array}\right\}$$

According to Galilean invariance, interactions expressed in either Σ or Σ' must be exactly the same and it follows that:

$$\boxed{\hat{V} = \hat{V}[(\hat{r}_1 - \hat{r}_2);\ (\hat{p}_1 - \hat{p}_2);\ (\hat{s}_1, \hat{s}_2);\ (\hat{t}_1, \hat{t}_2)]}$$

Notation. *Introduce the relative positions $\hat{r}_{12}$ and relative momenta $\hat{p}_{12}$:*

$$\hat{r}_{12} \equiv \hat{r}_2 - \hat{r}_1 \quad \text{and} \quad \hat{p}_{12} \equiv \hat{p}_2 - \hat{p}_1$$

then:

$$\boxed{\hat{V} = \hat{V}[\hat{r}_{12}, \hat{p}_{12}, (\hat{s}_1, \hat{s}_2); (\hat{t}_1, \hat{t}_2)]}$$

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Rotational Invariance: *It is assumed that our space is isotropic and thus any two reference frames that differ by orientation must be equivalent. This implies that interaction potential must be constructed out of scalars*

Examples: $\hat{r}_{12} \cdot \hat{r}_{12}, \quad \hat{p}_{12} \cdot \hat{p}_{12}, \quad \hat{r}_{12} \cdot \hat{p}_{12}, \quad \hat{r}_{12} \cdot \hat{s}_{12}, \quad \hat{p}_{12} \cdot \hat{s}_{12} \ \ldots$

Invariance Under Space Inversion: *Since strong interactions conserve the parity, Hamiltonian must depend on scalars and not pseudo-scalars:*

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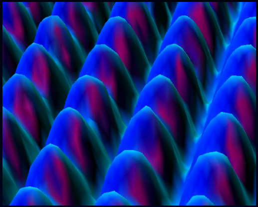
Time-Reversal Invariance: *We assume that the interaction Hamiltonian is time-reversal invariant. Recall:*

$$\hat{T}\hat{r}\hat{T}^{-1} = +\hat{r}, \quad \hat{T}\hat{p}\hat{T}^{-1} = -\hat{p}, \quad \hat{T}\hat{\ell}\hat{T}^{-1} = -\hat{\ell} \quad \text{and} \quad \hat{T}\hat{s}\hat{T}^{-1} = -\hat{s}$$

Hamiltonian must be contructed out of time-scalars, see a few examples:

$$\hat{p}_{12} \cdot (\hat{s}_1 + \hat{s}_2), \;\; (\hat{\ell}_1 + \hat{\ell}_2) \cdot (\hat{s}_1 + \hat{s}_2), \;\; (\hat{r}_{12} \wedge \hat{p}_{12}) \cdot (\hat{s}_1 + \hat{s}_2) \ldots$$

Training Our Imagination of Space & Interaction



- **Hamiltonians of fundamental interactions are invariant under the translations in any direction in any step**
- **Hamiltonians of fundamental interactions are invariant under any rotation through any angle**
- **They are simultaneously invariant under these and still some more transformations:**

How Many Symmetries Do You Know?

General Interaction Form Allowed by Symmetries

Conclusions about the N-N Force & Symmetries

- **Hamiltonians are: Hermitean and exchange-symmetric, Galilean symmetric, translation- and rotation-invariant, parity- and time-even**

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• The symmetries alone cannot determine the radial dependence of the interactions. For instance in a possible interaction operator

$$\hat{V}_{12} = v(r_{12})\, \hat{s}_1 \cdot \hat{s}_2$$

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• **Those functions must be determined by fitting to experimental results; fitting procedure remains phenomenological and not unique**

General Hamiltonian Form Allowed by Symmetries

Numerous experiments are compatible with the following forms of the Nucleon-Nucleon Interaction Hamiltonian

$\longrightarrow \longrightarrow \longrightarrow$

Fundamental Properties of Nucleon-Nucleon Forces

Let $\hat{x} \stackrel{df.}{=} \{\hat{r}, \hat{p}, \hat{s}, \hat{t}\}$. Nucleon-Nucleon interactions have the form:

$$\widehat{V}(\hat{x}_1, \hat{x}_2) \equiv \widehat{V}_C(\hat{x}_1, \hat{x}_2) + \widehat{V}_T(\hat{x}_1, \hat{x}_2) + \widehat{V}_{LS}(\hat{x}_1, \hat{x}_2) + \widehat{V}_{LL^2}(\hat{x}_1, \hat{x}_2)$$

where: C-central, T-tensor, LS-spin-orbit and LL^2-quadratic LS

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Central Interaction ($r_{12} \equiv |\vec{r}_1 - \vec{r}_2|$)

$$\begin{aligned} \widehat{V}_C(\hat{x}_1, \hat{x}_2) = V_0(r_{12}) \quad &+ \quad V_s(r_{12})\,[\hat{s}^{(1)} \cdot \hat{s}^{(2)}] \\ &+ \quad V_t(r_{12})\,[\hat{t}^{(1)} \cdot \hat{t}^{(2)}] \\ + V_{s-t}(r_{12})\,&[\hat{s}^{(1)} \cdot \hat{s}^{(2)}]\,[\hat{t}^{(1)} \cdot \hat{t}^{(2)}] \end{aligned}$$

Invariant under rotations, translations, inversion and time-reversal

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Tensor Interaction [Non-Central]

$$\vec{S}^{(12)} \overset{df.}{=} \frac{3\,(\hat{s}_1 \cdot \vec{r}_{12})(\hat{s}_2 \cdot \vec{r}_{12}) - (\hat{s}_1 \cdot \hat{s}_2)\, r_{12}^2}{r_{12}^2} \quad \textit{and} \quad r_{12} \overset{df.}{=} |\vec{r}_1 - \vec{r}_2|$$

$$\boxed{\widehat{V}_T(\hat{x}_1, \hat{x}_2) = [V_{t_0}(r_{12}) + V_{t_1}(r_{12})\, \hat{t}_1 \cdot \hat{t}_2]\, \hat{S}^{(12)}}$$

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Spin-Orbit Interaction [Non-Local]

$$\vec{L} \overset{df.}{=} \tfrac{1}{2}(\vec{r}_1 - \vec{r}_2) \wedge (\hat{p}_1 - \hat{p}_2), \quad r_{12} \overset{df.}{=} |\vec{r}_1 - \vec{r}_2| \; \textit{and} \; \hat{S} \overset{df.}{=} \hat{s}_1 + \hat{s}_2$$

$$\boxed{\widehat{V}_{LS}(\hat{x}_1, \hat{x}_2) = [V_{LS}^{t_0}(r_{12}) + V_{LS}^{t_1}(r_{12})\,\hat{t}_1 \cdot \hat{t}_2\,]\hat{L} \cdot \vec{S}}$$

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Quadratic Spin-Orbit Interaction [Non-Local]

$$\vec{L} \stackrel{df.}{=} \tfrac{1}{2}(\vec{r}_1 - \vec{r}_2) \wedge (\vec{p}_1 - \vec{p}_2) \quad \text{and} \quad r_{12} \stackrel{df.}{=} |\vec{r}_1 - \vec{r}_2|$$

$$\boxed{\widehat{V}_{LL}(\hat{x}_1, \hat{x}_2) = V_{LL}(r_{12})\{(\hat{s}_1 \cdot \hat{s}_2)\,\hat{L}^{\,2} - \tfrac{1}{2}[(\hat{s}_1 \cdot \hat{L})(\hat{s}_2 \cdot \hat{L}) + (\hat{s}_2 \cdot \hat{L})(\hat{s}_1 \cdot \hat{L})]\}}$$

Invariant under rotations, translations, inversion and time-reversal

Dynamics: What Is Doable and What Is Not?

- **Consider the motion of a system of $N = 100$ nucleons**

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$$\hat{H}(\underbrace{\hat{x}_1, \hat{x}_2, \; \dots \; \hat{x}_N}_{100 \times 12 = 1200 \;\; \text{operators}})\Psi = E\,\Psi$$

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$$\hat{H}(\underbrace{\hat{x}_1, \hat{x}_2, \; \dots \; \hat{x}_N}_{100\times 12=1200 \;\; \text{operators}})\Psi = E\,\Psi$$

Conclusion:

It is out of question to attack by brutal force...

But there exist helpful mechanisms - among others:

Symmetries and Spontaneous Symmetry Breaking

Spontaneous Symmetry Breaking - An Example

Given a system with symmetry $\{G\}$ i.e. $[H(\beta), G] = 0$. Here β is a parameter. Often a critical value, $\beta_{crit.}$, exists such that:

- For $\beta < \beta_{crit.}$ symmetry of solution is compatible with $\{G\}$
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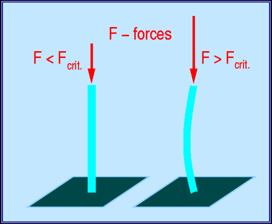
A Classical Example



Original system $\leftrightarrow$ axial symmetry

For $F > F_{crit.}$ we find infinitely many solutions at the same energy

Yet: The original axial symmetry will be *spontaneously* broken and only one among many directions - "privileged" !!!

Symmetry vs. Asymmetry - Historical Perspective

Caricature - A Classical (Macro) Example: Buridan's Donkey

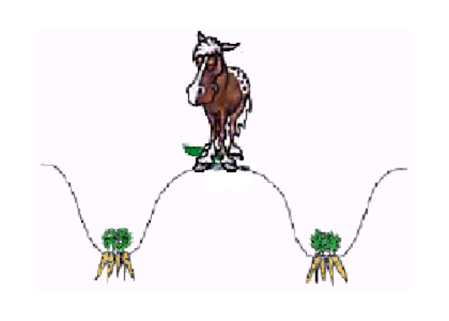


Figure: *Buridan's donkey, having two strictly identical bundles of carrots on both sides has no reason to select one of them, and dies of starvation*

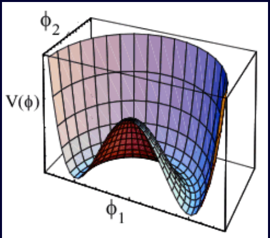
Symmetry: Spontaneously Broken

From preceding discussion we assume that the N-N interaction type

$$\widehat{V}(\hat{x}_1, \hat{x}_2) \equiv \widehat{V}_C(\hat{x}_1, \hat{x}_2) + \widehat{V}_T(\hat{x}_1, \hat{x}_2) + \widehat{V}_{LS}(\hat{x}_1, \hat{x}_2) + \widehat{V}_{LL^2}(\hat{x}_1, \hat{x}_2)$$

is invariant under rotations, translations, inversion and time-reversal

Spherical Symmetry?



The Nuclear Mean Field Theory ...

... is usually very successful. It is based on

$$\widehat{V}_{mf}(\hat{x}) = \int \psi^*(x')\widehat{V}(\hat{x}, \hat{x}')\psi(x')\, dx'$$

Some or all of the above symmetries will be broken by the mean-field Hamiltonian

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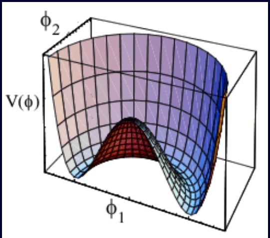

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Spontaneous Symmetry Breaking - An Illustration

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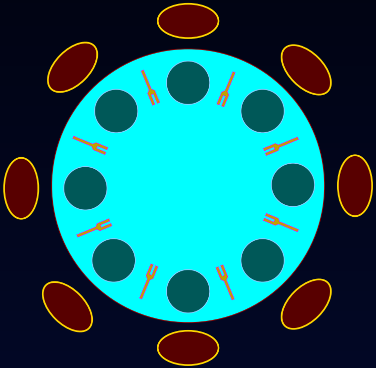

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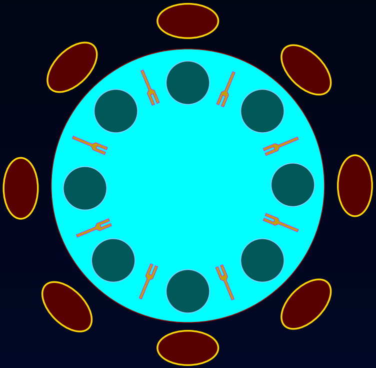

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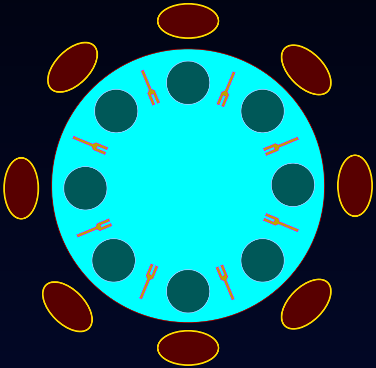


In such a case just listen, it is short but instructive ...

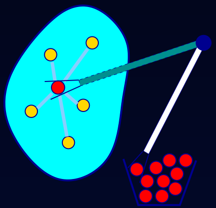
A Few Remarks about the Mean-Field Concept

- **A mean-field interaction can be seen as an algorithm probing the two-body interactions through a generalized weighted average $\widehat{V}$**

$$\boxed{\widehat{V}(\hat{x}) = \frac{1}{N-1}\sum_{j=1}^{(N-1)} \int dx_j \psi^*(x_j)\,\widehat{V}(\hat{x},\hat{x}_j)\,\psi(x_j)}$$

- **Obseve that summation implies the averaging over (N-1)-particles**
- **Notice also that the mean-potential $\widehat{V} = \widehat{V}(\hat{x})$ is a one-body operator**
- **Relativistic theory illustrated in the following provides a similar concept but using quantum field theory**

An N–Body System



Schematic: Probing 2-body interactions with an 'external' test-particle

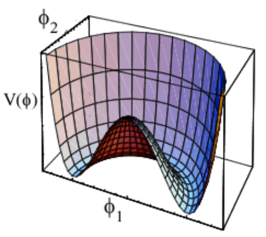
Symmetry: Exact, Approximate, Spontaneously Broken

From preceding discussion we assume that the N-N interaction

$$\widehat{V}(\hat{x}_1,\hat{x}_2) \equiv \widehat{V}_C(\hat{x}_1,\hat{x}_2) + \widehat{V}_T(\hat{x}_1,\hat{x}_2) + \widehat{V}_{LS}(\hat{x}_1,\hat{x}_2) + \widehat{V}_{LL^2}(\hat{x}_1,\hat{x}_2)$$

is invariant under rotations, translations, inversion and time-reversal

Spherical Symmetry?



The Nuclear Mean Field Theory ...

... is usually very successful. It is based on

$$\widehat{V}_{mf}(\hat{x}) = \int \psi^*(x')\widehat{V}(\hat{x},\hat{x}')\psi(x')\,dx'$$

Some or all of the above symmetries will be broken by the mean-field Hamiltonian

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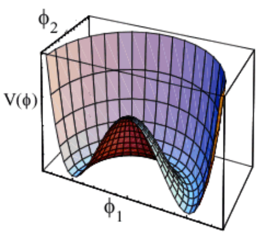

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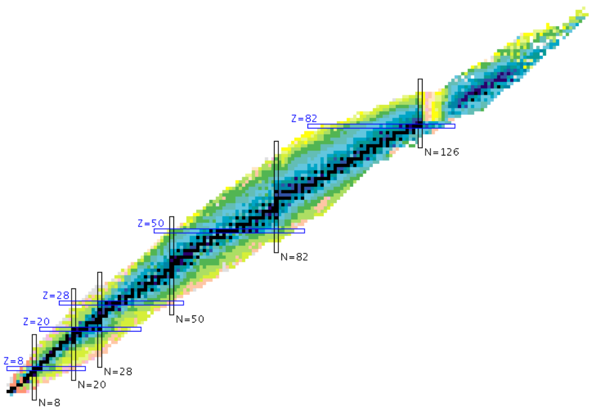
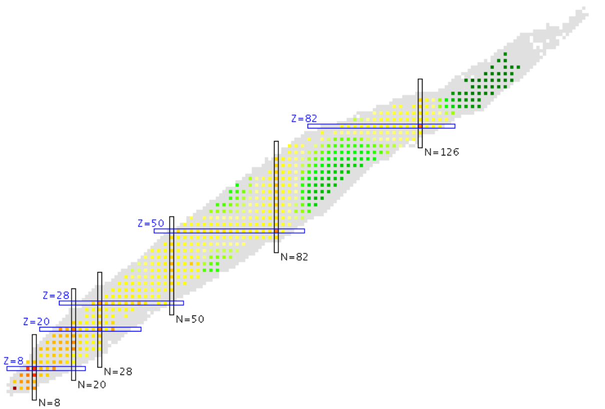
Nearly 3000 Systems Have Been Seen Experimentally



Among them, about two hundreds are stable; they are marked in black

In Majority of Them Spherical Symmetry is Broken



Among nearly 3000, more than 80% are strongly deformed

The Dominating Role of the Mean-Field Concept

A Few Important Conclusions:

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- The mean-field is by construction a one-body operator what implies significant simplifications

$$\boxed{\hat{H}_{\text{nature}}(\hat{x}_1, \hat{x}_2, \ \ldots \ \hat{x}_N) \approx \hat{H}_{\text{mean field}} = \sum_{i=1}^{N} \hat{h}(\hat{x}_i)}$$

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- From now on, effective theories can be constructed:

$$\boxed{\hat{H}_{nature} \approx \hat{H}_{mean\ field} + \hat{H}_{residual}}$$

More Precisely: Mean-Field and Residual Interactions

A Possible General Structure of Hamiltonians

- The unknown 'true' Hamiltonian is replaced by two effective ones

$$\hat{H}_{\text{nature}} \;\longrightarrow\; \hat{H} \approx \underbrace{\sum_{i=1}^{N} \hat{h}(\hat{x}_i)}_{\hat{H}_{\text{mf}}} + \underbrace{\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \hat{V}^{\text{res}}(\hat{x}_i \leftrightarrow \hat{x}_j)}_{\hat{H}_{\text{res}}}$$

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$$\hat{H}_{res} = \hat{V}_{pairing} + \hat{V}_{long\ range} + \hat{V}_{vib.coupling} + \dots$$

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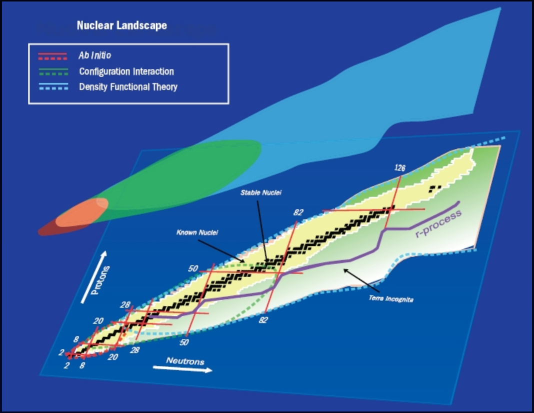
- In low-energy sub-atomic physics the theory calculations <u>without</u> considering the residual pairing are considered <u>not realistic</u>

$$\textbf{Pairing:} \quad \leftrightarrow \quad v^{\text{pairing}}_{\alpha\beta;\gamma\delta} \longleftarrow \textbf{to be defined}$$

Realistic Theories and their Today's Applicability



Present-day theories pretending to control ∼7000 nuclear systems, out of which more than a half is still to be produced

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Where are the symmetries?

Part V

Unitary Group U(n) and Symmetries of the Nuclear N-Body Hamiltonians

Unitary Groups and Infinitesimal Transformations

- Consider $x \in \mathbb{R}^n$ and transformations $x' = Ux$ for $U^\dagger U = \mathbb{1}$
- For infinitesimal transformations

$$U_\varepsilon \equiv \mathbb{1} + i\,\varepsilon\,S, \quad U^\dagger U = \mathbb{1} \to S = S^\dagger$$

- Consider $f = f(x)$ and infinitesimal transformation $x' = U_\varepsilon x$:

$$f' = f(x') = f(\{x_j + i\varepsilon \textstyle\sum_k S_{jk} x_k\}) = f(x) + i\,\varepsilon\, \textstyle\sum_{jk} S_{jk}\, x_k \partial_j f|_x$$

- Defining the generators as usual $\hat{g}_{jk} \equiv x_j \partial_k$ one has

$$\boxed{[\hat{g}_{\alpha\beta}, \hat{g}_{\gamma\delta}] = \delta_{\beta\gamma}\,\hat{g}_{\alpha\delta} - \delta_{\alpha\delta}\,\hat{g}_{\gamma\beta}}$$

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N-Body Systems with Two-Body Interactions

- Consider an N-particle system with a two-body Hamiltonian

$$\hat{\mathrm{H}} = \sum_{\alpha\beta} h_{\alpha\beta}\, c^+_\alpha c_\beta + \tfrac{1}{2} \sum_{\alpha\beta} \sum_{\gamma\delta} v_{\alpha\beta;\gamma\delta}\, c^+_\alpha c^+_\beta\, c_\delta\, c_\gamma$$

- Recall: For Fermions

$$\boxed{\{c^+_\alpha, c_\beta\} = \delta_{\alpha\beta}; \quad \{c^+_\alpha, c^+_\beta\} = 0 = \{c_\alpha, c_\beta\}; \quad \forall\, \alpha, \beta}$$

- Introduce operators $\hat{\mathrm{N}}_{\alpha\beta} \stackrel{df.}{=} c^+_\alpha c_\beta$

- It is easy to verify that

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U(n) Generators and the Many-Body Hamiltonian

- Let us trivially anticommute the operators

$$\hat{c}^+_\alpha\,(\hat{c}^+_\beta\,\hat{c}_\delta)\hat{c}_\gamma = (\hat{c}^+_\alpha\hat{c}_\delta)(\hat{c}^+_\beta\hat{c}_\gamma) - \delta_{\beta\gamma}\,(\hat{c}^+_\alpha\hat{c}_\gamma)$$

- Introduce the renormalised one-body term

$$\boxed{\langle\alpha|\hat{h}'_1|\beta\rangle = \langle\alpha|\hat{h}_1|\beta\rangle + \tfrac{1}{2}\textstyle\sum_\gamma^n\langle\alpha\gamma|\hat{h}_2|\beta\gamma\rangle}$$

- It follows that the Hamiltonian is a simple function of generators

$$\boxed{\hat{H} = \textstyle\sum_{\alpha\beta}^n\langle\alpha|\hat{h}'_1|\beta\rangle\;\hat{N}_{\alpha\beta} - \tfrac{1}{2}\textstyle\sum_{\alpha\beta}^n\sum_{\gamma\delta}^n\langle\alpha\beta|\hat{h}_2|\gamma\delta\rangle\;\hat{N}_{\alpha\delta}\;\hat{N}_{\beta\gamma}}$$

- Interactions $\leftrightarrow$ matrices; unitary group formalism $\leftrightarrow$ generators

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$$L_0\,\psi_p = m\psi \quad \text{and} \quad L_o\,(L_\pm\,\psi) = (m\pm 1)\,L_\pm\,\psi \quad \text{m=weight}$$

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• Using the maximum-weight polynomial with $m_{max} = l$ we obtain the bases $\{\psi_{lm}\}$ of irreducible representations of $O(3)$

Comment about Group Representations: Case U(n)

- Consider homogeneous, independent polynomials of the order p:

$$\Psi \equiv \prod_{\alpha=1}^{n} (\hat{c}_{\alpha}^{+})^{p_\alpha} \quad \text{with} \quad \sum_{\alpha=1}^{n} p_\alpha = p; \quad p_\alpha \geq 0: \ \text{integer}$$

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$$\hat{N}_{\alpha\beta} \equiv \hat{c}_\alpha^+ \hat{c}_\beta \quad \text{one has} \quad \hat{N}_{\alpha\alpha}\, \Psi|0\rangle = p_\alpha\, \Psi|0\rangle; \quad \alpha = 1, 2, \; \ldots \; n$$

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- From the commutation relations

$$\boxed{[\hat{N}_{\alpha\alpha}, \hat{N}_{\alpha\beta}] = +\hat{N}_{\alpha\beta} \quad \text{and} \quad [\hat{N}_{\beta\beta}, \hat{N}_{\alpha\beta}] = -\hat{N}_{\alpha\beta}} \qquad \text{follows:}$$

$$\begin{aligned}
\hat{N}_{\alpha\alpha}(\hat{N}_{\alpha\beta}\,\Psi) &= (p_\alpha + 1)\,\hat{N}_{\alpha\beta}\Psi, \quad \alpha < \beta: \quad \text{weight-raising} \\
\hat{N}_{\beta\beta}(\hat{N}_{\alpha\beta}\,\Psi) &= (p_\alpha - 1)\,\hat{N}_{\alpha\beta}\Psi, \quad \alpha > \beta: \quad \text{weight-lowering}
\end{aligned}$$

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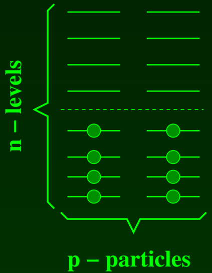
• Dimensions of the irreducible representation (n, p):

$$C^n_p = n!/p!(n-p)!$$

i.e. the number of combinations for p-particles on the n-levels



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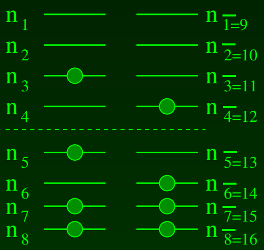
- Moreover, under the condition:

$$\sum_j n_j = p, \quad \text{for} \quad n_j = 0 \text{ or } 1$$

each state can be seen as an integer that corresponds to its binary representation

$$E = \sum_{k=1}^{n} b_k 2^{k-1} \rightarrow |0010101100010111\rangle$$



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In other words:

The whole 'technology' is set for an effective computer work using e.g. shell-model techniques while keeping the link with the group-theory through the explicit presence of the group generator matrices

N-Body Hamiltonians and U(n)-Group Generators

- N-Body Hamiltonians are functions of U_n-group generators

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- Two-body interactions lead to quadratic forms of $\hat{\mathrm{N}}_{\alpha\beta} = c^+_\alpha c_\beta$, three-body interactions to the cubic forms of $\hat{\mathrm{N}}_{\alpha\beta}$, etc.
- Hamiltonians of the N-body systems can be diagonalised within bases of the irreducible representations of unitary groups
- Solutions can be constructed that transform as the U_n-group representations thus establishing a link $\mathrm{H} \leftrightarrow U_n$-formalism and, most importantly, block-diagonal structure of the Hamiltonian, labelling the states with the good quantum numbers etc.

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$$
\begin{aligned}
\hat{\mathrm{H}} = \sum_{\alpha\beta} h_{\alpha\beta}\,\hat{N}_{\alpha\beta} \quad &+ \quad \frac{1}{2!}\sum_{\alpha\beta}\sum_{\gamma\delta} v_{\alpha\beta;\gamma\delta}\,\hat{N}_{\alpha\gamma}\hat{N}_{\beta\delta} \\
&+ \quad \frac{1}{3!}\sum_{\alpha\beta}\sum_{\gamma\delta}\sum_{\kappa\rho} u_{\alpha\beta\kappa;\gamma\delta\rho}\,\hat{N}_{\alpha\gamma}\hat{N}_{\beta\delta}\hat{N}_{\kappa\rho} \;\ldots
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Consequences in Terms of Sub-Groups

Group U_n has numerous sub-groups: U_m and SU_m with $m < n$, similarly O_m, SO_m and in particular R_3 and all the point groups

• The subgroup chains, properties of the Casimir operators give the mathematical framework to study - among others:

1. **The so called 'Dynamical symmetries'**
2. **Accidental degeneracies, etc.**

Remark:

There is no unique definition of the 'dynamical symmetry'; various authors use the liberty of stressing various aspects - we will here more on the subject later in the week

Subgroup Structure Can Be Very, Very Rich ...

32 Point Groups: Subgroups

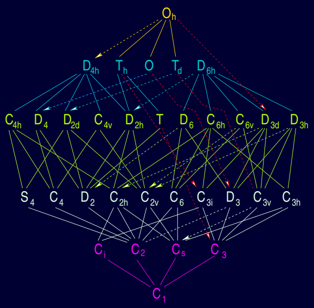


Figure: *Richness of the sub-group structures at the end of chain...*

Dashed lines indicate thatsub-groups marked are not invariant

The trivial groups are denoted as follows:

$$C_1 \equiv \{\mathbb{I}\},\ C_s \equiv \{\mathbb{I}, \hat{\sigma}\},$$
$$C_i \equiv \{\mathbb{I}, \hat{\pi}\},\ C_2 \equiv \{\mathbb{I}, \hat{R}_2\},$$
$$C_3 \equiv \{\mathbb{I}, \hat{R}_3, \hat{R}_3^2\}$$

Here we show the structure only at the very end of the U_n chain - this helps imagining how rich the full group structure is ...

If Tetrahedral Nuclei Have No Q_2-Moments ...

At the exact tetrahedral symmetry the Q_2 moments possibly vanish

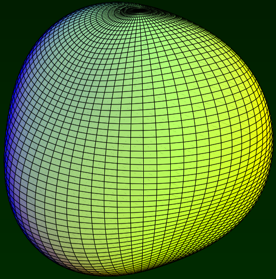
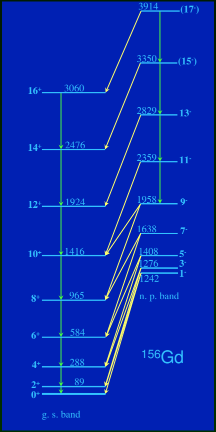


Figure: *Equilibrium shape* $t_1 = 0.15$



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Experiments at Jyvaskyla, Legnaro & Argonne with the Gammasphere were performed, others are being prepared

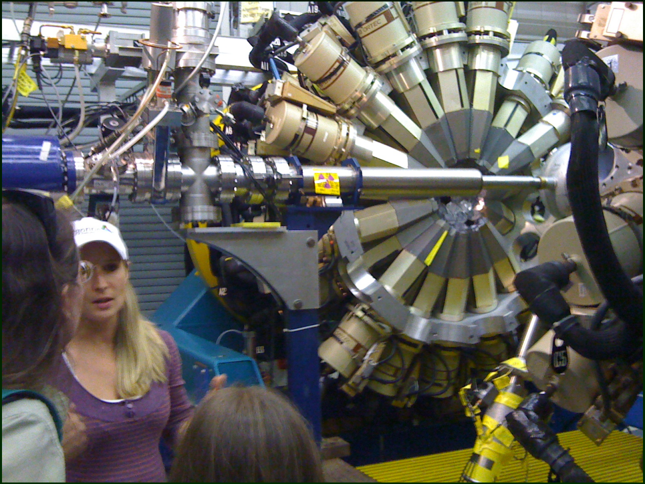

Figure: *Gammasphere is the world's most powerful gamma-spectrometer (surrounded by the very friendly, warm, stimulating, American atmosphere)*

Part VI

From Nuclear to Sub-Nuclear Particles, Degrees of Freedom and Symmetries

Discovering Sub-Atomic Particles: The Pion

- Following the prediction of Hideki Yukawa of 1935...

The Discovery of the Pion

- In 1947: *Charged pions* $\pi^{\pm}$ are discovered by Cecil Powell, César Lattes and Giuseppe Occhialini at the University of Bristol
- In 1949: Nobel Prize awarded to H. Yukawa, for predicting the existence of mesons
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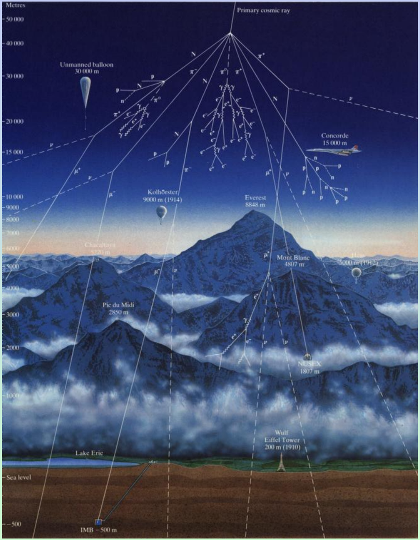
Information that Is Just Falling from Heaven



First Particles Called Strange: Strangeness

- **The discovery of the neutral, strange Λ-particle, in 1951**

Here: An example of the results from liquid hydrogen bubble chamber* at Brookhaven National Laboratory. The yellow line at the bottom is an incoming high-energy proton, it collides with a proton at rest in the liquid hydrogen creating many particles. Seven positive pions, a proton, and a positive kaon (shown in red) curve off to the right, while seven negative pions (blue) move to the left. A neutral Λ is also produced which travels upwards undetected and then decays into a proton (yellow) and a negative pion (purple). NB: the green curve at the bottom is due to an electron which has been knocked out of its orbit by the passing proton.

*Credits: Brookhaven National Laboratory

$$\Lambda \longrightarrow \pi^- + p$$

π^- - purple, p - yellow

Even More Strange Particles Discovered Soon After

- *These new particles can be grouped; within the group they decay very fast, but very slowly to the outside of the group - where from their name*
- *By attributing a new quantum number*, 'strangeness' S, we are able to systematise their decay and reaction properties (Table for $s = \frac{1}{2}$ particles)*

Symbol	S	⟨Life-time⟩ sec	Q	Decay
Λ^0	-1	2.6×10^{-10}	0	$\begin{cases} p+\pi^- \\ n+\pi^0 \end{cases}$
Σ^+	-1	8.0×10^{-11}	$+1$	$\begin{cases} p+\pi^0 \\ n+\pi^+ \end{cases}$
Σ^0	-1	7.4×10^{-20}	0	$\Lambda^0+\gamma$
Σ^-	-1	1.4×10^{-10}	-1	$n+\pi^-$
Ξ^0	-2	2.9×10^{-10}	0	$\Lambda^0+\pi^0$
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- *These new particles can be grouped; within the group they decay very fast, but very slowly to the outside of the group - where from their name*

- *Moreover, strange particles are produced always in pairs in the strong interactions of the non-strange hadrons for instance* $\pi^+ + p \longrightarrow K^+ + \Sigma^+$

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Summarising the Results of the First Discoveries

• There exist 'heavy' particles such as nucleons (fermions), 'medium heavy' mesons (bosons) and 'light' particles (fermions)

• Names: for nucleons, mesons and other heavy particles

$$\boxed{\textbf{Baryons - from Greek: } \beta\alpha\rho\upsilon\zeta = \textbf{ heavy}}$$

and for the light particles:

$$\boxed{\textbf{Leptons - from Greek: } \lambda\epsilon\pi\tau\upsilon\varsigma = \textbf{ delicate}}$$

• Experiments show that baryons and mesons interact, create other particles and decay in very short times comparable with 10^{-24} secs

• We identify the short-time processes with the strong interactions; nucleons and mesons belong to this category

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Elementary Sub-Nuclear Particles

Baryons, Mesons and Partons

• **We may observe that baryons are, on average, heavier than mesons thus they may contain more really elementary constituents (partons)**

• If we wish to keep simplicity: the smallest number of elementary constituents must be 2 (mesons) and one bigger must be 3 (baryons)

• If we attribute the baryonic 'charge' to all the baryons $B = +1$, then anti-baryons must have $B = -1$ and all other particles $B = 0$

• It follows that elementary constituents must have $B = \frac{1}{3}$ so that baryons may have $B = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

• ... while mesons must be composed of pairs: parton - anti-parton and thus $B = \frac{1}{3} + \frac{\bar{1}}{3} = \frac{1}{3} - \frac{1}{3} = 0$

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- **If we wish to keep simplicity: the smallest number of elementary constituents must be 2 (mesons) and one bigger must be 3 (baryons)**

- **If we attribute the baryonic 'charge' to all the baryons $B = +1$, then anti-baryons must have $B = -1$ and all other particles $B = 0$**

- **It follows that elementary constituents must have $B = \frac{1}{3}$ so that baryons may have** $B = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

- **... while mesons must be composed of pairs: parton - anti-parton and thus** $B = \frac{1}{3} + \frac{\bar{1}}{3} = \frac{1}{3} - \frac{1}{3} = 0$

- **Elementary charges of partons must be a multiple of $Q_{el.} = \frac{1}{3}$ e**

- **Partons must be Fermions so that both half-integer and integer spins can be constructed**

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From Baryons, Mesons and Partons to Quarks

- **The smallest number of partons is 2; the simplest interaction law assures that the interactions do not depend on the type of parton**

$$\begin{bmatrix} u' \\ d' \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \times \begin{bmatrix} u \\ d \end{bmatrix} \leftrightarrow \text{SU(2)-symmetry}$$

- **We must introduce the electric charges by convention:**

Quark	Symbol	Spin	B	Q
up	u	1/2	1/3	+2/3
down	d	1/2	1/3	−1/3

- **Partons with these properties are called quarks (see below)**
- **Test for the nucleons and pions**

$$\boxed{p = uud, \quad n = udd, \quad \pi^+ = u\bar{d}, \quad \pi^- = d\bar{u}, \quad \pi^0 = u\bar{u} \;\; \text{and/or} \;\; d\bar{d}}$$

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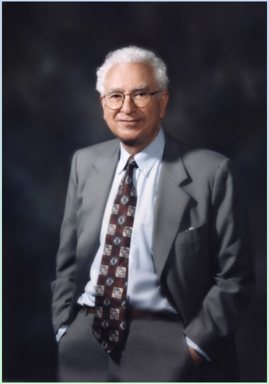
Historical Remarks: Parton and Quark Models

- **Quark Model, 1964 by Murray Gell-Mann and George Zweig**

The quark model uses the concept of quarks with several properties just as introduced above (see also below).

The initial reaction of the physics community to the proposal was mixed. There was particular contention about whether the quark was a physical entity, or an abstraction used to explain certain new concepts that were not well understood at the time.



Murray Gell-Man

Historical Remarks: Parton and Quark Models

- **Parton Model formulated in 1969 by Richard P. Feynman**

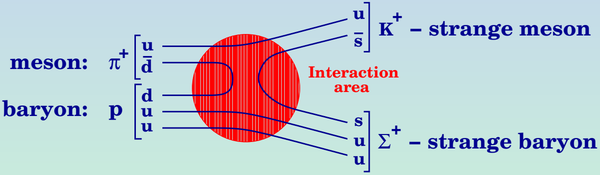
In this model, a hadron is composed of a number of point-like constituents, called "partons". Additionally, the hadron is in a reference frame where it has infinite momentum - a valid approximation at high energies.

Quark model can be seen as a particular realisation of the parton model.

Richard P. Feynmann

Strange Particles: Extension of the Quark Model

• The conservation of strangeness could not be accounted for with the presence of two quarks only, where from the new hypothesis of the existence of the third ('strange') quark s

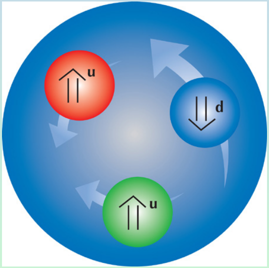


• Illustration of the process of quark - anti-quark annihilation in a central 'interaction area'. It can be viewed in analogy to the other annihilation processes such as $e^+ + e^- \rightarrow 2\gamma$ and many others

Nucleon Spin & Orbital Motion of Quarks

Jefferson National Accelerator Facility, Virginia, USA. Report of a discovery that the spins of the proton's two up quarks (u) are aligned parallel to the overall spin of the proton, but the same is not true for the proton's down quark (d)

In order to make the experimental data on quark spin agree with theory, the authors had to take into account the once-neglected orbital motion of quarks inside the proton



Credits: Jefferson Lab. and Zheng et al., Phys. Rev. Lett. 2003

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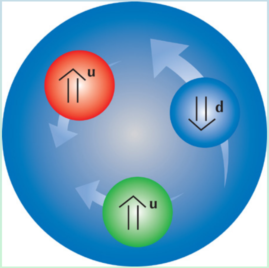

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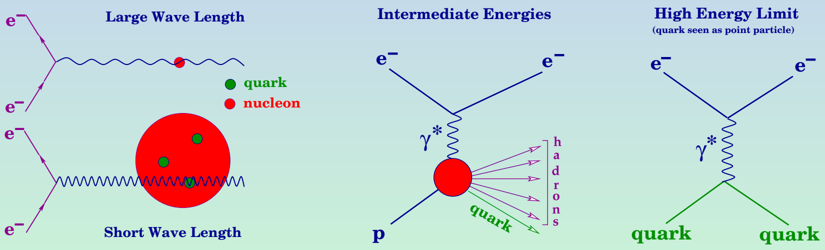
Today's Truly Elementary Particles: Quarks

Principle of Experimental Tests of the Quark Model

- Probing the quark structure of protons through deep inelastic scattering of high-energy electrons; the quark structure is resolved through the virtual photons when $\lambda \ll 1$ Fm



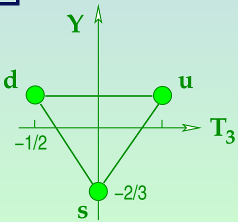
- Experiments on high energy $e + p$ scattering fully confirmed these qualitative considerations providing the basis for the quark model

Convenient Observables: Hypercharge, Y, Isospin T

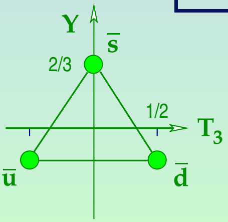
Quark	Spin	B	Q	T_3	S	Y
u	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	0	$\frac{1}{3}$
d	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	$\frac{1}{3}$
s	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	-1	$-\frac{2}{3}$

$$\boxed{Y \overset{df}{=} B + S}$$

$$\boxed{Q = T_3 + \tfrac{1}{2} Y}$$



Quarks



Anti–Quarks

Historical Remarks about SU(3)-Symmetry

- One shows that generators of the $SU(3)$-group can be taken as

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- There are 3 independent SU(2) subgroups with generators:

$$\{\lambda_1, \lambda_2, x\}, \quad \{\lambda_4, \lambda_5, y\} \ \text{and} \ \{\lambda_6, \lambda_7, z\}$$

where the x, y, z are linear combinations of the diagonal λ_3 and λ_8

Historical Remarks about SU(3)-Symmetry

• One may define $\boxed{\hat{T}_3 \equiv \lambda_3}$ and $\boxed{\hat{Y} \equiv 2/\sqrt{3}\lambda_8}$ as well as

$$\boxed{\hat{\mathbf{T}}_\pm \equiv \lambda_1 \pm \mathbf{i}\,\lambda_2; \quad \hat{\mathbf{V}}_\pm \equiv \lambda_4 \pm \mathbf{i}\,\lambda_5; \quad \hat{\mathbf{U}}_\pm \equiv \lambda_6 \pm \mathbf{i}\,\lambda_7}$$

• One finds easily

$$[\hat{\mathbf{T}}_+, \hat{\mathbf{T}}_-] = 2\hat{\mathbf{T}}_3; \quad [\hat{\mathbf{U}}_+, \hat{\mathbf{U}}_-] = 2\hat{\mathbf{U}}_3; \quad [\hat{\mathbf{V}}_+, \hat{\mathbf{V}}_-] = 2\hat{\mathbf{V}}_3$$

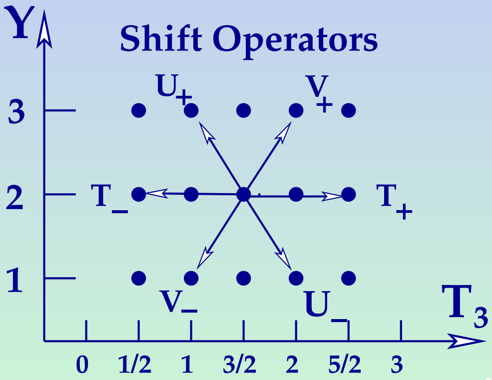
with

$$2\hat{\mathbf{U}}_3 \equiv \tfrac{3}{2}\hat{\mathbf{Y}} - \hat{\mathbf{T}}_3 \;\text{ and }\; 2\hat{\mathbf{V}}_3 \equiv \tfrac{3}{2}\hat{\mathbf{Y}} + \hat{\mathbf{T}}_3$$

• These are the so-called T-, U- and V-spins of Gell-Mann; the maximum number of commuting operators is 2 for instance $[\hat{T}_3, \hat{Y}] = 0$

U-Spin, V-Spin, T-spin and SU(3)-Symmetry

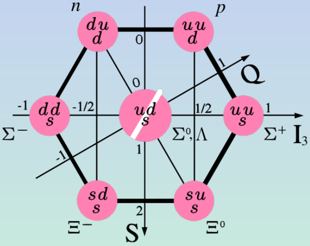
- Action of the shift operators on the common basis states $|Y, T_3\rangle$

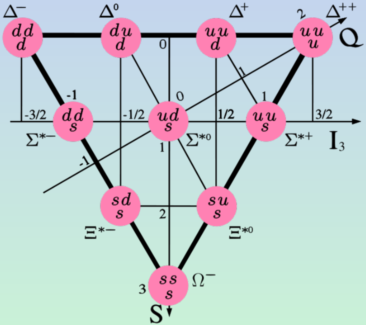


- These simple constructions lead to the octet and nonet diagrams of the baryons and mesons paving the way for the $SU(3)$ symmetry in QCD

Three Quarks and Resulting Baryon Periodic Tables

• Combining u, d and s quarks and using Y (alternatively* S) vs. isospin T_3 representation we obtain octet and decuplet structures





• The predictions of the existence of all these particles have been confirmed experimentally supporting the idea of quark constituents

*Since $Y \overset{df}{=} B + S$, one can use alternatively S; indeed $B = const.$ implies constant shift in this case

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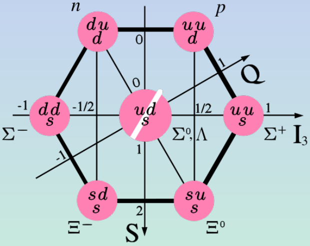

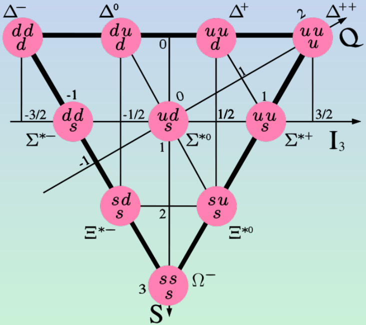

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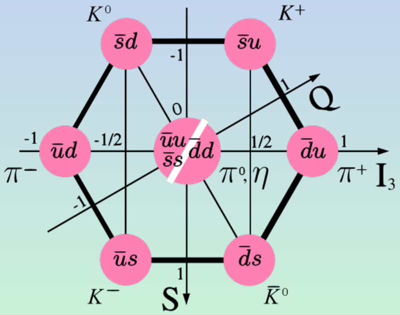
Three Quarks and Resulting Meson Periodic Tables

• Combining quark - anti-quark pairs and using the strangeness S vs. isospin T_3 representation we obtain a nonet structure



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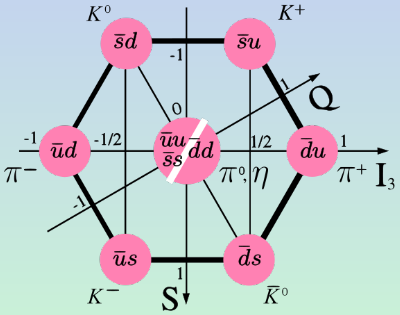

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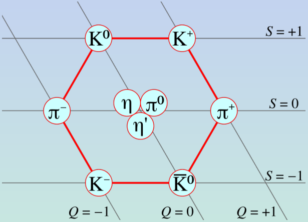
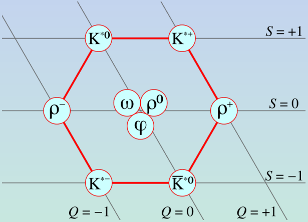
• Quark - anti-quark pairs implies $s = 0$ and $s = 1$ meson nonets



Spin s=0 Pseudo-Scalar Mesons



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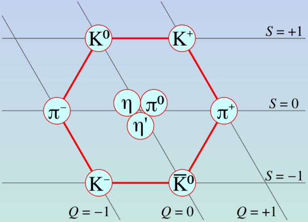

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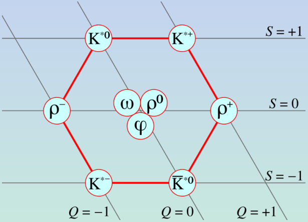

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Part VII

The Heaviest Quarks

Elementary Constituents of Matter - Today

- **This is a subject about high-energy limit - Perhaps some other time?**

November Revolution: J and ψ that Became J/ψ

- The existence of the fourth quark has been predicted, among others, by Glashow, Iliopoulos and Maiani, in 1970

- On the 14^{th} of November 1974 a discovery of a new particle has been announced simultaneously by Stanford Linear Accelerator Center (SLAC) and Brookhaven National Laboratory (BNL) groups

- The SLAC group called the new particle ψ and the BNL called it J - both discoveries concerned the same particle

- The particle (the only one named with two letters) was called J/ψ - and the leaders* of the teams obtained Nobel prize in 1976

- The new particle is interpreted today as a pair new-quark new-anti-quark, the former called ‘charm’, c: thus $J/\psi \leftrightarrow c\bar{c}$

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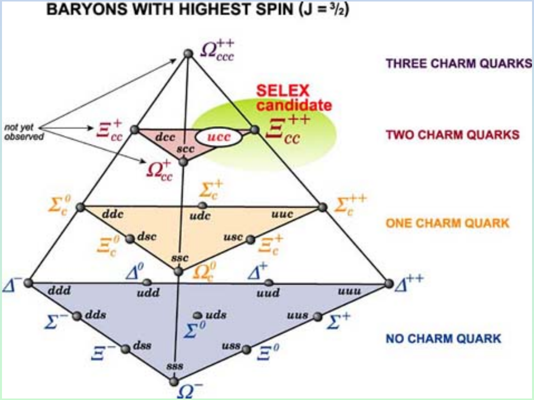
A New Version of the ‘Particle Periodic Table’

- Baryons with increasing number of charmed quarks (counting from the bottom to the top of the figure)



The Heaviest Elementary: Top and Bottom Quarks

• Precision calculations of Gerardus 't Hooft and Martinus Veltman predict the existence of yet another quark, called top, t

• After these predictions the top quark anti-quark pair was discovered in 1995 at Fermilab (Tevatron) by CDF and D0 collaborations

• Nobel Prize for Gerardus 't Hooft and Martinus Veltman in 1999

• Single quark production via weak interactions: in March 2009, both CDF and D0 announced discovery of a single-top production

• According to Standard Model t-lifetime is $\sim 1 \times 10^{-25}$ sec, about 20 times shorter than the time-scale for strong interactions - therefore quark t does not hadronize

• **Top t offers a unique opportunity to study a "bare" quark**

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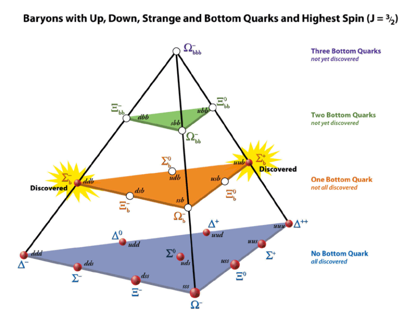
• **Top t offers a unique opportunity to study a "bare" quark**

From Mendeleiev's to Baryon Periodic Table

Baryons are particles made of three quarks. The particles can exist in a ground state (J=1/2) and an excited state (J=3/2). This figure shows the various three-quark combinations with J=3/2 that are possible using the three lightest quarks – up, down and strange – and the bottom quark. Past experiments discovered all of the baryons made of light quarks. The CDF discovery is the first observation of baryons with one bottom quark and spin J=3/2.



Collider Detector at Fermilab (CDF). The discovery of the positively charged Σ_b^+ and the negatively charged Σ_b^- in both spin configurations.

Credits: Fermi Lab. Press Release

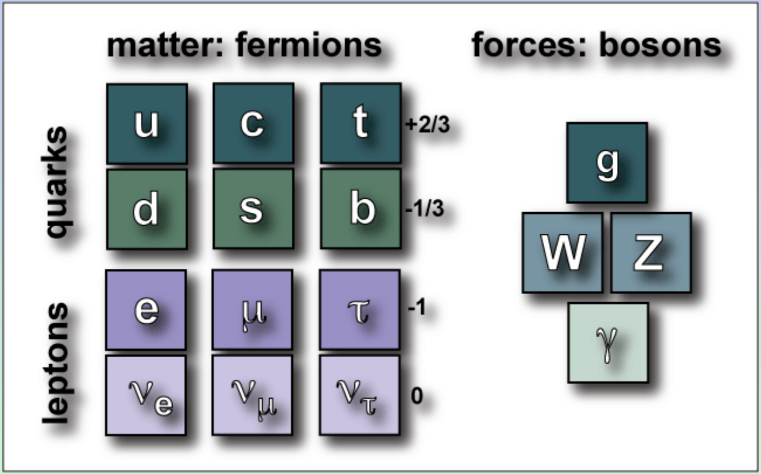
Summary of Quark Flavour Properties

- **The meaning of some symbols: J-spin, B baryon-number, Q-charge, T_z-isospin, C-charmness, S-strangeness, T-topness, B′-botomness**

Name	Symb	M MeV/c^2	J	B	Q	T_z	C	S	T	B′
Up	u	1.5 to 3.3	$\frac{1}{2}$	$\frac{1}{3}$	$+\frac{2}{3}$	$+\frac{1}{2}$	0	0	0	0
Down	d	3.5 to 6.0	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	0	0	0
Charm	c	1 270	$\frac{1}{2}$	$\frac{1}{3}$	$+\frac{2}{3}$	0	+1	0	0	0
Strange	s	104	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	−1	0	0
Top	t	171 200	$\frac{1}{2}$	$\frac{1}{3}$	$+\frac{2}{3}$	0	0	0	+1	0
Bottom	b	4 200	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	0	−1

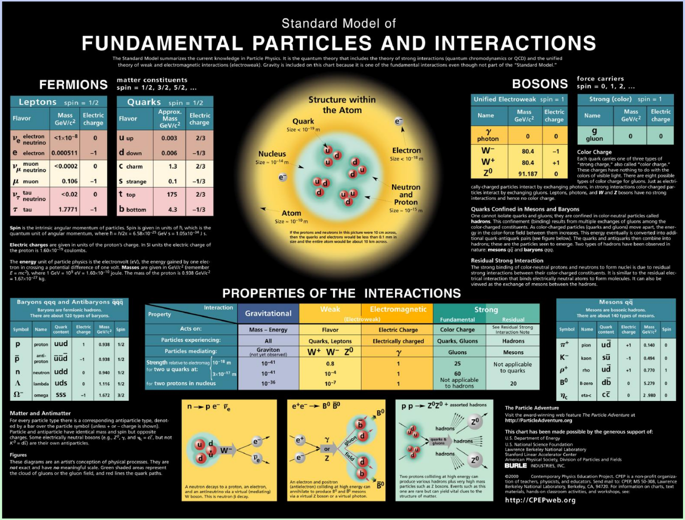
• Quarks are considered point particles; • A quark of one flavour can transform into a quark of another flavor only through the weak interaction; • These transitions occur through emission of virtual W bosons

Quarks, Leptons and Force-Transmitting Bosons



You came here from the beginning of Part V. Wish to return?

Key Issues in the Standard Model



Part VIII

Quantum Relativistic Wave Equation, Related Symmetries

Dirac's Search for the Relativistic Wave Equation

- Dirac constructs his equation that describes relativistic $s = \frac{1}{2}$ particles and admits probabilistic interpretation:

$$\boxed{(i\hbar\gamma^\mu \hat{p}_\mu - m_0 c)\psi(x) = 0; \;\; \{\gamma^\mu, \gamma^\nu\} = 2 \cdot \mathbb{I}\, g^{\mu\nu}, \;\; 4 \times 4 \text{ matrices}}$$

- Solutions ψ, Dirac spinors also called bi-spinors, have the structure

$$\psi = \binom{\xi}{\eta}, \;\; \xi = \binom{\xi_1}{\xi_2}, \;\; \eta = \binom{\eta_1}{\eta_2}; \;\; \gamma^0 = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{bmatrix}, \;\; \gamma^j = \begin{bmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{bmatrix}$$

Notation: $x \equiv \{x^\mu\} = \{ct, \vec{r}\}, \;\; p \equiv \{p^\mu\} = \left\{\frac{E}{c}, \vec{p}\right\}, \;\; \bar{\psi} \equiv \psi^\dagger \gamma^0$

- Dirac demonstrated the searched conservation of probability

$$\boxed{\hat{j}^\mu \stackrel{df}{=} \bar{\psi}\,\gamma^\mu \psi, \;\; \partial_\mu \hat{j}^\mu = 0} \qquad j^0 = \bar{\psi}\,\gamma^0 \psi = \psi\,\gamma^0 \gamma^0 \psi = \psi^\dagger \psi \geq 0$$

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Dirac's Problem with the Relativistic Wave Equation

- The problem: Dirac equation admits negative energy solutions

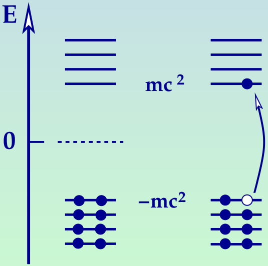
$$\boxed{\psi^{(\pm)} = \binom{\xi}{\eta}_o \exp\left\{\mp i\, p\, x/\hbar\right\} \;\text{ with }\; E = \pm\sqrt{(c\vec{p})^2 + (m_0 c^2)^2}}$$

- These solutions forced a bit artificial Dirac sea interpretation: infinitely many particles that form permanent 'vacuum'

- On the other hand it allowed for the pair-creation mechanism:

- The hole of a given-charge appears as one opposite charge particle;

- The corresponding excitation energy is always positive, masses remain equal



Discovery of Anti-Electrons: Matter and Anti-Matter

• In 1932: Paul Adrien Maurice Dirac, after a series of difficulties with the negative energy solutions to the Dirac equation, postulates the existence of a positron, anti-particle associated with electron

• In 1932: Carl David Anderson, Swedish American finds positrons, electron-positron pairs, using gamma rays produced by the natural radioactive nuclides: particle-anti-particle production

• In 1911-1913: Victor Francis Hess, American of Austrian origin, discovers that radiation detected at 5 km above the sea level is twice as high as at the sea level (cosmic origin)

• In 1936: Nobel Prize for C. G. Anderson (anti-electrons) and V. F. Hess (cosmic rays) for their discoveries

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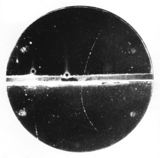
Discovery of Anti-Electrons: Historical Documents



The discovery of the positron in 1932 by Carl Anderson studying cosmic rays. The particle was deflected by a magnetic field in the opposite direction to the electron, but was too light to be a proton

*This bubble chamber photograph shows an electron and a positron (anti-electron) that are spiralling in opposite directions**

** Credits: CERN*

Particles and Anti-Particles: a New Symmetry

As a physicist - Whenever you see such an image - recall: particle-antiparticle symmetry and doubling the universe. In physics: charge conjugation

Some Historical Steps in Retrospection

• The free Dirac equation generalises simply, within the so-called minimal coupling scheme, for electromagnetic interactions

$$\boxed{p_\mu \to \left(p_\mu - \tfrac{e}{c}A_\mu\right) : \;\Rightarrow\; \left[\gamma^\mu(\hat{p}_\mu - \tfrac{e}{c}A_\mu) - m_0 c\right]\psi(x) = 0}$$

• We can introduce a charge conjugation operator, $\hat{\mathcal{C}}$, transforming a given solution ψ into opposite-charge same-mass solutions ψ_c:

$$\hat{\mathcal{C}} = \hat{\mathcal{C}}(\{\gamma^\mu\}) \;\;\to\;\; \hat{H}_c = \hat{\mathcal{C}}\,\hat{H}\,\hat{\mathcal{C}}^{-1} \text{ and } \psi_c = \hat{\mathcal{C}}\,\psi$$

• With the help of the charge conjugation operation one can show

$$\boxed{\langle\psi_c|\hat{H}|\psi_c\rangle = -\langle\psi|\hat{H}|\psi\rangle \;\text{ and }\; \langle\psi_c|\hat{p}|\psi_c\rangle = -\langle\psi|\hat{p}|\psi\rangle}$$

i.e. the negative-energy particles seen as positive-energy anti-particles moving in the opposite sense of time

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Some Historical Steps in Retrospection (2)

- Similarly to the concept of charge conjugation $\hat{\mathcal{C}}$, the other discrete symmetries such as inversion $\hat{\mathcal{P}}$ and time-reversal $\hat{\mathcal{T}}$ have been introduced

- This led to the discoveries of the $\hat{\mathcal{C}}\hat{\mathcal{P}}$ as well as $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}$ symmetries and consecutively to the discovery of partial $\hat{\mathcal{C}}\hat{\mathcal{P}}$ symmetry breaking

- All these concepts were developed further by Richard Feynman while constructing quantum electrodynamics (QED); Nobel Prize in 1965, together with Julian Schwinger and Sin-Itiro Tomonaga

- Using field theory formalism the concept of charge-conjugation has been generalised to other charges such as baryonic, leptonic ... etc.

- These new concepts helped in constructing modern theories of elementary particles (see below)

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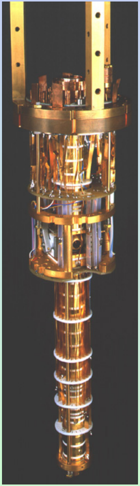
Producing Anti-Matter: Today

Producing Anti-Hydrogen: ATRAP Experiment

- Basing on his general equation governing the motion of relativistic Fermions, Dirac has formulated for the first time the prediction of the existence of antiparticles

- This prediction opened the way to the idea that each particle has an anti-particle partner, not just electrons

- In this way we arrive at the hypothesis of Doubling the Forms of Matter: *There exists Matter and Anti-Matter*



To the right: The ATRAP* apparatus combines positrons (which enter from the top) with anti-protons (which enter from below) and meet about one-third of the way up from the bottom to make neutral anti-hydrogen atoms. To do this the positrons pass through a special rotable electrode (the element with the circular hole near the bottom of the wide part of the apparatus)

***Credits: Physical Review Letters, November 2002**

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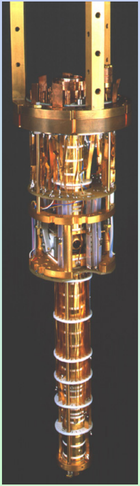

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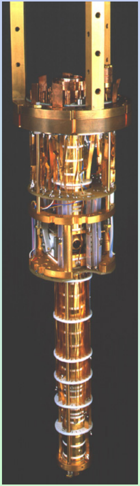

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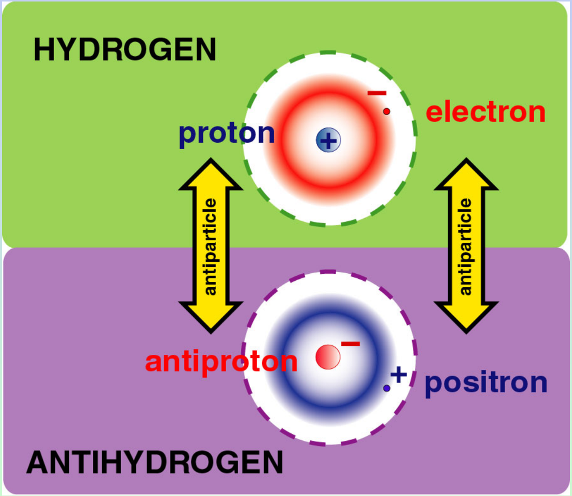
Anti-Hydrogen and Its Twin Brother: Portraits



Part IX

A Short Descriptive Lesson about QCD

Lagrangian Densities and Field Equations

- Remarks about the notations (four-vectors, metric tensor, scalar)

$$dx \equiv \{c\,dt, d\vec{x}\};\; dx^0 = c\,dt;\; dx^1 = dx;\; dx^2 = dy;\;\; dx^3 = dz$$

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag.}\{+1,-1,-1,-1\};\; dx^2 = g_{\mu\nu}\,dx^\mu dx^\nu = c^2\,dt^2 - d\vec{x}^{\,2}$$

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- Lagrangian: Classical mechanics compared to classical field theory

$$\boxed{\mathbf{L = L(q, \dot{q}; t)} \ \leftrightarrow\ \boldsymbol{\mathcal{L} = \mathcal{L}(\{\psi^\alpha\}, \{\psi^\alpha_{,\mu}\}; x)};\ \ \boldsymbol{\psi^\alpha_{,\mu} \stackrel{\text{df}}{=} \frac{\partial \psi^\alpha}{\partial x^\mu}}}$$

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- Action integral of the Lagrangian density defined as Lorentz-scalar

$$\boxed{\mathcal{S} \overset{\mathrm{df}}{=} \int \mathbf{d^4x}\, \mathcal{L}(\{\psi^\alpha\}, \{\psi^\alpha_{,\mu}\}; \mathbf{x})}$$

Lagrangian Densities and Field Equations

- Applying variational ('least action') principle gives field equations

$$\text{Euler-Lagrange} \;\longrightarrow\; \boxed{\frac{\partial}{\partial \mathsf{x}^{\mu}}\left(\frac{\partial \mathcal{L}}{\partial \psi^{\alpha}_{,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial \psi^{\alpha}} = 0} \;\longleftarrow\; \text{Field-Equations}$$

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we obtain the usual form of the 'generalised' current conservation

$$\boxed{\partial_\mu J^\mu = 0 \quad \text{where} \quad J^\mu \equiv L^\mu_\alpha\,(\delta'\psi^\alpha) + \left(\delta^\mu_\nu\,\mathcal{L} - \psi^\alpha_{,\nu}\,L^\mu_\alpha\right)\delta x^\nu}$$

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$$\boxed{\partial_\mu \mathbf{J}^\mu = 0 \;\text{ where }\; \mathbf{J}^\mu \equiv \mathbf{L}^\mu_\alpha\,(\delta'\psi^\alpha) + \left(\delta^\mu_\nu\,\mathcal{L} - \psi^\alpha_{,\nu}\,\mathbf{L}^\mu_\alpha\right)\delta \mathbf{x}^\nu}$$

- Defining a physical problem means: Define the Lagrangian density

An Example of a Classical Gauge Theory

- Consider n scalar fields ψ_a with common mass m. The action is:

$$\mathcal{S} = \int d^4x \sum_{a=1}^{n} \left[\tfrac{1}{2}\,\partial_\mu\psi_a\,\partial^\mu\psi_a - \tfrac{1}{2}\,m^2\,\psi_a^2\right]; \quad \delta\mathcal{S} \longrightarrow \underline{\underline{\text{K-G Equations}}}$$

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- Consider a group of orthogonal transformations so that $\mathcal{O} \in SO_n$

$$\boxed{\mathcal{O}^\mathsf{T}\mathcal{O} = \mathbb{1}:\; \Phi \longrightarrow \Phi' \equiv \mathcal{O}\Phi; \;\; (\partial_\mu\Phi) \longrightarrow (\partial_\mu\Phi)' = \mathcal{O}(\partial_\mu\Phi)}$$

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- **Note:** Gauge symmetry implies an existence of conservation laws

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- Finally, the classical fields (Lagrangian) will need to be quantised

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- According to the interpretation of the *Interaction Lagrangian*, it describes particles interacting via the exchange of the gauge bosons

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- Consider the electrons in classical electrodynamics. The Action is:

$$\mathcal{S} = \int \bar{\psi}(i\hbar c\, \gamma^\mu \partial_\mu - mc^2)\psi \, \mathrm{d}^4 x$$

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- The gauge-field $A_{\mu}(x)$ becomes four-vector potential of the E-M

$$\mathcal{L}_{\text{int}} = \frac{e}{\hbar}\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x) \;\to\; \mathcal{L}_{\text{QED}} = \bar{\psi}(i\hbar c\, \gamma^{\mu} D_{\mu} - mc^2)\psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}$$

Quantum Chromodynamics - The Principal Actors

• Compared to one-type of particles in QED - in QCD there are 6 quark fields denoted by ψ_n, each one being a Dirac four-spinor $[\psi_\mu]_n$

• In QCD, indices $i \in [1, 6]$ are related to the so-called quark flavours - they have given names *up, down, charm, strange, top* and *bottom*

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• There are eight generators of this group (Gell-Mann matrices) $\hat{T}_a$

$$\left[\hat{\mathbf{T}}_\mathbf{a}, \hat{\mathbf{T}}_\mathbf{b}\right] = \mathbf{i}\,\mathbf{C}^\mathbf{c}_\mathbf{ab}\,\hat{\mathbf{T}}_\mathbf{c}; \quad \mathbf{a}, \mathbf{b}, \mathbf{c} = \mathbf{1}, \mathbf{2}, \ldots \mathbf{8}$$

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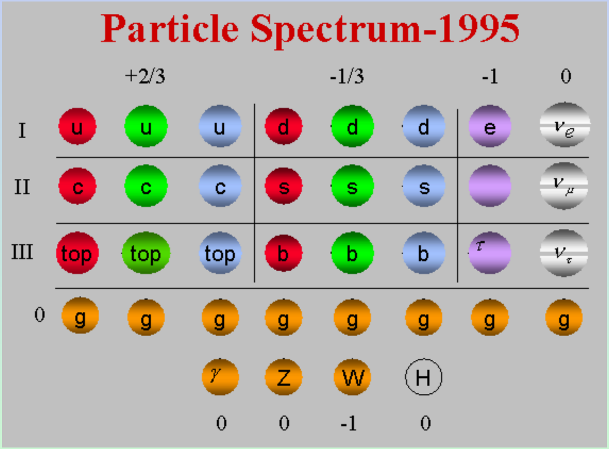
• Quarks appear in 3 variants numbered with 3 indices, given the names *red, blue* and *green*, the latter corresponding to the indices of the basis vectors of the (3D) fundamental representation of $SU(3)$

Table of Quarks with Colours



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- One can show that the QCD Lagrangian density is the following

$$\mathcal{L} = -\underbrace{\tfrac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu}_a}_{(A)} - \underbrace{\textstyle\sum_{i=1}^n \bar{\psi}_i\,\gamma^\mu\big(\partial_\mu - ig\,A^a_\mu\,\hat{T}_a\big)\psi_i}_{(B)} - \underbrace{\textstyle\sum_{i=1}^n m_i\,\bar{\psi}_i\psi_i}_{(C)}$$

(A) - describes free gluon fields expressed with four-potentials A^a_μ

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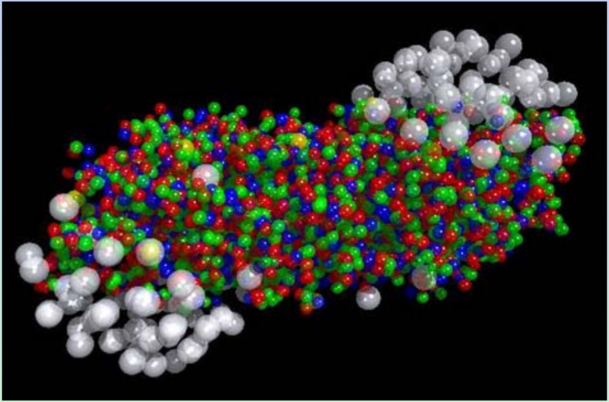
(C) - describes the ‘free’ six QCD quarks with the rest-masses $\mathbf{m_i}$

Quantum Chromodynamics - Quarks and Nucleons



High energy collisions - red, green and blue quarks - an artist view